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Almost-Riemannian Geometry from a Control Theoretical Viewpoint

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Contents

Introduction

The purpose of this thesis is to study a generalization of Riemannian geometry that naturally arises in the framework of control theory. A Riemannian distance on a smooth surface M can be seen as the minimum-time function of an optimal control problem where admissible velocities are vectors of norm one. The control problem can be written locally as

$$\dot{q} = uX(q) + vY(q), \quad u^2 + v^2 \leq 1, \tag{1}$$

where $\{X, Y\}$ is a local orthonormal frame. Almost-Riemannian structures generalize Riemannian ones by allowing X and Y to be collinear at some points. In this case the corresponding Riemannian metric has singularities, but under generic conditions the distance is well-defined. For instance, if the two generators satisfy the Hörmander condition, system (1) is completely controllable and the minimum-time function still defines a continuous distance on the surface.

Our work is devoted to the local and global analysis of this kind of metric structures using a control theoretical approach. More precisely, an almost-Riemannian structure (ARS) on a surface M is a triple $(E, f, \langle \cdot, \cdot \rangle)$, where E is a Euclidean bundle of rank two over M (i.e. a vector bundle whose fibre is equipped with a smoothly-varying scalar product $\langle \cdot, \cdot \rangle_q$) and $f : E \to TM$ is a morphism of vector bundles such that the evaluation at q of the Lie algebra generated by the submodule

$$\Delta = \{f \circ \sigma \mid \sigma \text{ section of } E\}$$

is equal to $T_q M$ for every $q \in M$. The Euclidean structure $\langle \cdot, \cdot \rangle$ on E induces a symmetric positive definite bilinear form $G(\cdot, \cdot)$ on the submodule Δ. At points q where $f|_{E_q}$ is an isomorphism G acts as a tensor, i.e., $G(V, W)|_q$ depends only on $V(q), W(q)$. This is no longer true at points belonging to the singular set $\mathcal{Z} = \{q \in M \mid \dim f(E_q) < 2\}$, which is generically a smooth embedded submanifold of dimension one. Clearly, an ARS is Riemannian if and only if f is an isomorphism of vector bundles or, equivalently, the singular set is empty. The singularity of the metric tensor does not prevent from defining geodesics: classical methods of optimal control theory, mainly based on the Pontryagin Maximum Principle (see [7]), allow to show that geodesics are well-defined and do not have singularities.

The simplest example of genuinely almost-Riemannian structure is provided by the Grushin plane, which is the ARS on $\mathbb{R}^2$ where $E = \mathbb{R}^2 \times \mathbb{R}^2$, $f((x, y), (a, b)) = ((x, y), (a, xb))$, and $\langle \cdot, \cdot \rangle$ is the canonical Euclidean structure on $\mathbb{R}^2$. In this case a global orthonormal frame is given by $X(x, y) = \partial_x$, $Y(x, y) = x\partial_y$ and the singular set is indeed nonempty, being equal to the y-axis. This example is named after V.V. Grushin who studied in [27, 28] analytic properties of the operator $\partial_x^2 + x^2 \partial_y^2$ and of its multidimensional generalizations (see also [23]).

Metric structures defined globally by a pair of vector fields on a surface (not necessarily parallelizable) arise naturally in the context of quantum control (see [14, 13]). Indeed, consider the ARS on $S^2 \subset \mathbb{R}^3$ where E is the trivial bundle of rank two over S^2 and the image under f of a global orthonormal frame for $\langle \cdot, \cdot \rangle$ on E is the pair $X(x, y, z) = (y, -x, 0)$, $Y(x, y, z) = (0, z, -y)$. Then the two generators are linearly dependent on the intersection of the sphere with the plane $\{y = 0\}$
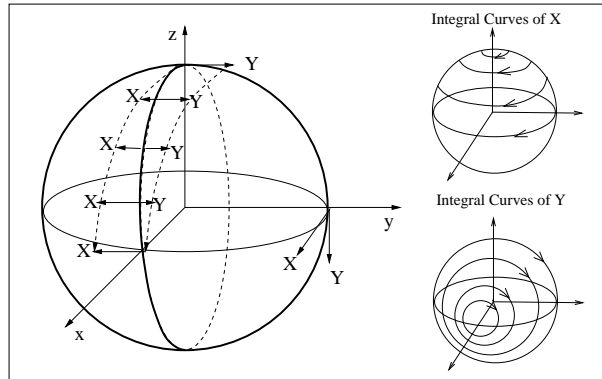
Figure 1: Almost-Riemannian structure on the 2-sphere

(see Figure 1). In this model, the sphere represents a suitable state space reduction of a three-level quantum system and the orthonormal generators X and Y are the infinitesimal rotations along two orthogonal axes, modeling the action on the system of two lasers in the rotating wave approximation.

The first important work studying general properties of ARSs is [3] where the authors provide the characterization of generic ARSs by means of local representations, that are pair of vector fields given by the push-forward of a local orthonormal frame along a coordinate system (see Figure 2). This result essentially states that for points of a generic almost-Riemannian surface three possibilities arise: ordinary points, Grushin points, and tangency points. At ordinary points the distribution, i.e., the evaluation at points of the submodule Δ, is two-dimensional and the structure is Riemannian. At Grushin points the distribution is one-dimensional and transversal to the singular set, but it is sufficient to add to the distribution a Lie bracket between two elements of a local orthonormal frame to get the whole tangent plane. At tangency points the distribution is still one-dimensional but it is tangent to the singular set and a bracket of length two between the two elements of a local orthonormal frame is needed to span the whole tangent plane at the point. Generically, the tangency points are isolated.

ARSs present very interesting phenomena. For instance, from the local point of view, the relations between curvature and conjugate points change, as the presence of a singular set permits the conjugate locus to be nonempty even if the Gaussian curvature is negative, where it is defined. This can be easily seen computing the geodesic flow at a point of the Grushin plane. In this case the exponential map at any point can be computed explicitly, via the Pontryagin Maximum Principle and the curvature is given by $K(x,y) = -2/x^2$. It turns out that there are geodesics (starting from an ordinary or a Grushin point) that reach a conjugate point in finite time after crossing the singular set. From the global point of view, the relations between curvature and topology of the surface change as well. Indeed, another result proved in [3] is an extension of the Gauss–Bonnet theorem for orientable ARSs on orientable surfaces, under the hypothesis that there are not tangency points (an ARS is orientable if the vector bundle E is orientable). The analogous version for
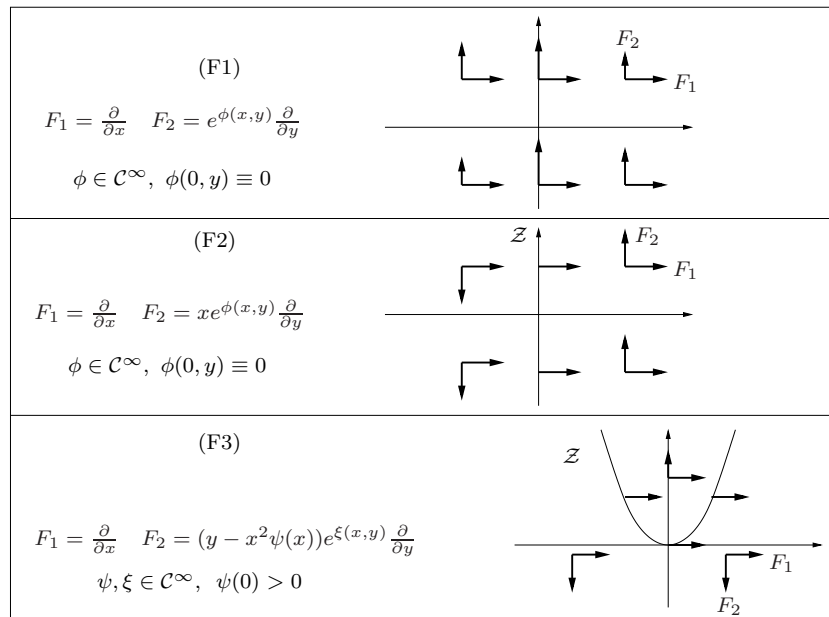
Figure 2: The local representations established in [3]

domains with boundary was later given in [17]. Other generalizations of the Gauss–Bonnet formula can be found in [4] for contact sub-Riemannian manifolds and in [34, 35] for pseudo-Riemannian manifolds.

This thesis is mainly devoted to the analysis of almost-Riemannian structures with tangency points.

In Chapter 2 we study ARSs from a global point of view. The first important result amounts to prove a relation among the topology of the almost-Riemannian surface and the Euler number $e(E)$ of the vector bundle associated with the structure (see [6]). The theorem states that for a generic oriented ARS on a compact oriented surface M the relation

$$\chi(M^+) - \chi(M^-) + \tau = e(E) \tag{2}$$

holds, where τ is the number of revolutions of Δ along $\mathcal{Z}$ (see Section 2.1.1), χ denotes the Euler characteristic and the subsets $M^\pm$ are defined as follows. Once the orientations on E and on M are fixed, M is split into two open sets M^+, M^- such that $\mathcal{Z} = \partial M^+ = \partial M^-$, $f : E|_{M^+} \to TM^+$ is an orientation preserving isomorphism and $f : E|_{M^-} \to TM^-$ is an orientation reversing isomorphism. Since the Euler number identifies a vector bundle in the class of oriented rank-two vector bundles over a compact oriented surface, this theorem classifies generic oriented ARS on compact oriented surfaces in terms of the associated vector bundle. The strategy to prove (2) is based on the construction of a section of the sphere bundle of E on a tubular neighborhood of $\mathcal{Z}$ having as singularities the set of tangency points. Such section is then extended to the complement in the surface of a finite set and the

sum of the indexes at all singularities is showed to be equal to $\chi(M^+) - \chi(M^-) + \tau$. As a direct consequence we get that an ARS is trivializable, i.e., E is isomorphic to the trivial bundle, if and only if $\chi(M^+) - \chi(M^-) + \tau = 0$. This generalizes and provides the converse result of a fact proved in [3] stating that if tangency point are absent (thus $\tau = 0$) and the structure is trivializable then $\chi(M^+) - \chi(M^-) = 0$. An alternative proof of the fact that the condition $\chi(M^+) - \chi(M^-) + \tau = 0$ is sufficient for the structure to be trivializable can be found in [5].

The second theorem we show is a generalization of the Gauss–Bonnet formula in [3] to the case where tangency points are present (see [6]). To this aim, the first task is to generalize the notion of integrability of the curvature given in [3]. Let dA_s be the two-form on $M \setminus \mathcal{Z}$ obtained as the pushforward along f of a given volume form for the Euclidean structure on E. In [3] the integrability of the curvature with respect to the ARS was defined through the existence and finiteness of the limit

$$\lim_{\varepsilon \searrow 0} \int_{M_\varepsilon} K \, dA_s, \tag{3}$$

where $M_\varepsilon = \{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}$ and $d(\cdot, \cdot)$ is the almost-Riemannian distance. If M has no tangency point, the limit in (3) was shown to exist and to be equal to $2\pi(\chi(M^+) - \chi(M^-))$. It turns out that the hypothesis about the absence of tangency points is not just technical. Indeed, we present some numerical simulations strongly hinting that the limit in (3) diverges, in general, if tangency points are present. One possible explanation of this fact is the interaction between different orders in the asymptotic expansion of the almost-Riemannian distance. To avoid this interference, we define a 3-scale integral of the curvature depending on the choice at each tangency point of a smooth curve transversal to the distribution. Thanks to the canonical choice for such curve given in Chapter 3, the new notion of integrability is still intrinsic. Moreover, it coincides with the one in [3] if the set of tangency point is empty. Using the Gauss–Bonnet formula for domains with boundary [17] and the classification of ARSs given above, we show that for a generic oriented ARS on a compact oriented surface the 3-scale integral of the curvature is equal to $2\pi e(E)$. When the almost-Riemannian structure is trivializable, the integral of the curvature vanishes, the Euler number of the trivial bundle being zero. Once applied to the special subclass of Riemannian structures, such a result simply states that the integral of the curvature of a parallelizable compact oriented surface (i.e., the torus) is equal to zero. In a sense, in the Riemannian case the topology of the surface gives a constraint on the total curvature through the Gauss–Bonnet formula, whereas for an almost-Riemannian structure induced by a single pair of vector fields the total curvature is equal to zero and the topology of the manifold constrains the metric to be singular on a suitable set.

The last part of Chapter 2 is devoted to the description of how the presence of the singular set and, in particular, of tangency points affect the distance associated with the ARS. Namely, we focus our attention on the problem of Lipschitz equivalence among different almost-Riemannian distances. Despite of the Riemannian case where all distances on the same compact oriented surface are Lipschitz equivalent, the classification of almost-Riemannian distances is finer. We show that the Lipschitz equivalence class of a 2-ARS is determined by the topology of the sets M^+

and M^- and by the location of tangency points with their contributions, which are integers in $\{\pm 1\}$ (see Section 2.1.1). It turns out that all the information needed to identify the Lipschitz equivalence class of an almost-Riemannian distance can be encoded in a labelled graph that is naturally associated with the structure. The vertices of such graph correspond to connected components of $M \setminus \mathcal{Z}$ and the edges correspond to connected components of $\mathcal{Z}$. The edge corresponding to a connected component $\mathcal{W}$ of $\mathcal{Z}$ joins the two vertices corresponding to the connected components of $M \setminus \mathcal{Z}$ adjacent to $\mathcal{W}$. Every vertex is labelled with its orientation (± 1 if it is a subset of $M^{\pm}$) and its Euler characteristic. Every edge is labelled with the ordered sequence of signs (modulo cyclic permutations) given by the contributions at the tangency points belonging to $\mathcal{W}$. We say that two labelled graphs are equivalent if they are equal or they can be obtained by the same almost-Riemannian structure reversing the orientation of the vector bundle. Theorem 2.9 states that two almost-Riemannian distances on the same compact oriented surface are Lipschitz equivalent if and only if the ARSs have equivalent graphs (see [16]). However, notice that in general Liptschitz equivalence does not imply isometry. Indeed, the Lipschitz equivalence between two structures does not depend on the metric structure but only on the submodule Δ. This is highlightened by the fact that the graph itself depends only on Δ.

In Chapter 3 we focus on local aspects of ARSs. The geometry of the nilpotent approximation of a generic ARS at a tangency point was studied in [2, 12]. In those papers the exponential map of the nilpotent approximation at a tangency point was computed in terms of Jacobi elliptic functions. The first step of our work consists in using this optimal synthesis to compute the jet of the exponential map at a tangency point of a generic ARS. This allows us to estimate the conjugate locus and the cut locus in the generic case (see [11]). In particular, we show that if a relation among the coefficients in the Taylor expansions of the functions appearing in a local representation holds, then the cut locus accumulates as an asymmetric cusp (see Figure 3).

Another question we address is how to find a local representation at tangency points which is completely reduced, in the sense that it depends only on the ARS. Remark that the local representations given in Figure 2 corresponding to Riemannian and tangency points are not completely reduced. Indeed, there exist changes of coordinates and rotations of the frame for which an orthonormal frame has the same expression as in (F1), respectively (F3), but with a different function ϕ, respectively with different functions ψ and ξ. The construction of the coordinate systems for which the local expressions (F1), (F2), (F3) apply is based on the choice of a smooth parametrized curve everywhere transversal to the distribution. If such a curve can be built canonically, then one gets a local representation that cannot be further reduced. For Riemannian points, a canonical parametrized curve transversal to the distribution can be easily identified, at least in the generic case, as the level set of the curvature. For Grushin points, a canonical curve transversal to the distribution is the singular set which has also a natural parameterization. As concerns the local expression (F3), in [3] the choice of the smooth parametrized curve is arbitrary and not canonical. Our concern is to find a canonical one, that is, to

identify the true invariants of the structure at a tangency point. The most natural candidate for such a curve is the cut locus from the tangency point. Nevertheless, this is not a good choice, since in general the cut locus starting from the point is not smooth (see Figure 3). Another possible candidate is the cut locus from the
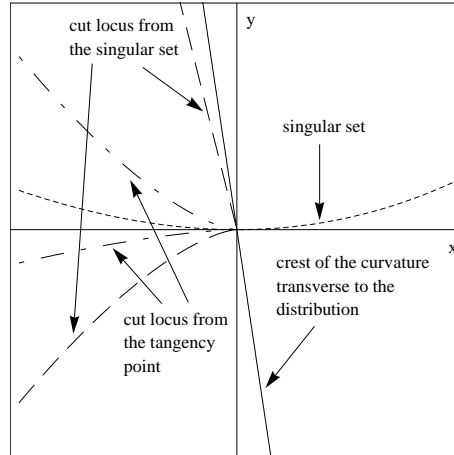

Figure 3: The singular set (dotted line), the cut locus from a tangency point (semi-dashed line), the cut locus from the singular set (dashed line), and the crest of the Gaussian curvature (solid line) transversal to the distribution for the ARS with orthonormal frame $F_1 = \frac{\partial}{\partial x}$, $F_2 = (y - x^2 - x^3)\frac{\partial}{\partial y}$

singular set in a neighborhood of the tangency point. We show that in general this locus is not smooth either (see figure 3). This is essentially due to the fact that the distance from $\mathcal{Z}$ has different orders depending on the side from which we approach the singular set. A third possibility is to look for crests or valleys of the Gaussian curvature which intersect transversally the singular set at the tangency point. Theorem 3.4 consists in the proof of the existence of such a crest (see [15]). Moreover, this curve admits a canonical regular parameterization. Then, a completely reduced local representation is obtained implicitly by requiring this curve to be the vertical axis. Explicit relations between the Taylor coefficients of the functions ψ and ξ at the point can be further obtained.

Chapter 4 provides a general method to find normal forms for a special class of vector fields at non-isolated singular points (see [25]). Such method allows to find normal forms for direction fields (i.e., equivalence classes of vector fields modulo multiplication by a never-vanishing smooth function) corresponding to geodesic flow of metrics with singularities such as pseudo-Riemannian, Klein-type and almost-Riemannian metrics on surfaces. We study metrics with singularities on $\mathbb{R}^2$ given by

$$ds^2 = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2 \qquad (4)$$

where the coefficients a, b, c characterize each case. We consider first pseudo-Riemannian metrics, characterized by the quadratic form in (4) being positive definite on

an open domain, degenerate on the smooth curve $\{b^2 - ac = 0\}$, and undefinite on some other open domain. In particular, we study the geodesic flow at points of the curve $\{b^2 - ac = 0\}$. We then study the geodesic flow at points of the y-axis for metrics (4) of Klein type where $a = \bar{a}/x^{2n}, b = \bar{b}/x^{2n}, c = \bar{c}/x^{2n}$, $n \in \mathbb{N}$ and $\bar{a}, \bar{b}, \bar{c}$ are smooth functions such that $\bar{b}^2 - \bar{a}\bar{c} > 0$. Finally, we consider the geodesic flow at Grushin points for generic almost-Riemannian metrics on $\mathbb{R}^2$. Here we consider a Lagrangian approach to the geodesic problem, differing from the Hamiltonian one presented in Chapter 3. In each of the three types of metrics above, the Lagrangian is smooth at all points except for a regular hypersurface.

These examples motivate the study of a particular class of vector fields on a open subset of $\mathbb{R}^n$ which have a singularity of divide-by-zero type. We establish some general facts about vector fields of the form

$$W(x) = \frac{1}{f(x)^r} V(x), \quad x \in \mathbb{R}^n, r > 0, \tag{5}$$

(where V is a smooth vector field and f is smooth scalar function vanishing on a regular hypersurface in $\mathbb{R}^n$) under two special assumptions that allow to infer the phase portraits of V and W. Many variational problems in differential geometry and calculus of variations (such as the geodesic problem for the three kind of metrics with singularities given above) are characterized by Lagrangian (or Hamiltonian) functions that are smooth at all points except for a regular hypersurface. The vector field corresponding to the Euler–Lagrange equations of such problems is divergence-free and takes the form (5). We prove some simple theorems about vector fields of the form (5) under some special assumptions (fulfilled, for instance, by divergence-free vector fields) without any additional hypothesis on V. In particular, these results show the key role of singular points of the field V in the applications and provide a relation among the eigenvalues of the linearization of V at a singular point belonging to the surface $\{f = 0\}$. The application of the theory of normal form for vector fields at non-isolated singular points (see Section 4.2 and references therein) to the vector field V in (5) leads to find normal forms for the direction fields corresponding to geodesic flows on surfaces with the three types of singular metrics mentioned above. Unfortunately, this method cannot be applied to the geodesic flow at tangency points of ARSs, where the problem of finding a normal form is more intricated, as we discuss in Chapter 3. The reason is that all the eigenvalues of the linearization of V at tangency point vanish, whence a normal form for V has not been found yet.

List of publications

The researches presented in this thesis appears in the following publications.

(a)   A.A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, M. Sigalotti, *Two-dimensional almost-Riemannian structures with tangency points*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27, no. 3, pp.793–807, 2010.

(b)   B. Bonnard, G. Charlot, R. Ghezzi, G. Janin, *The sphere and the cut locus at a tangency point in two-dimensional almost-Riemannian geometry*, Jour. Dyn. Con. Syst., to appear, hal-00517193

(c)   U. Boscain, G. Charlot, R. Ghezzi, *A normal form for generic 2-dimensional almost-Riemannian structures at a tangency point*, preprint 2010, arXiv:1008.5036

(d)   U. Boscain, G. Charlot, R. Ghezzi, M. Sigalotti, *Lipschitz classification of almost-Riemannian distances on compact oriented surfaces*, preprint 2010, arXiv:1003.4842

(e)   R. Ghezzi, A.O. Remizov, *On a class of vector fields with discontinuity of divide-by-zero type and its applications*, preprint 2010, arXiv:1007.0912

(cp)   A.A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, M. Sigalotti, *Two-dimensional almost-Riemannian structures with tangency points*, Proc. of the 48th IEEE Conference on Decision and Control, December 16-18, 2009. Shangai, China

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Almost-Riemannian structures in optimal control theory

This chapter is a brief survey of some of the basic notions and facts in the framework of sub-Riemannian geometry and it is aimed at fixing the notations that will be used throughout chapters 2 and 3. Remark that, far from being self-contained, we skip all the proofs of the results, for which we refer the reader to [7, 9]. In Section 1.2 we recall the basic result (as well as its proof) in [3] providing local representations for generic almost-Riemannian structures.

1.1 Rank-varying sub-Riemannian structures

Unless specified, smooth means $\mathcal{C}^\infty$-smooth, manifolds are smooth and without boundary; vector fields and differential forms are smooth. Given a vector bundle E over a manifold M, the set of smooth sections of E, denoted by $\Gamma(E)$, is naturally endowed with the structure of $\mathcal{C}^\infty(M)$-module. In the case $E = TM$ we denote $\Gamma(E)$ by $\mathrm{Vec}(M)$.

We start by defining rank-varying distributions on a manifold.

Definition 1.1. A *rank-varying distribution* on a n-dimensional manifold M is a pair (E, f) where E is a vector bundle of rank k over M and $f : E \to TM$ is a morphism of vector bundles, i.e., f is a smooth map such that **(i)** if $\pi : TM \to M$ and $\pi_E : E \to M$ denote the canonical projections, the diagram (1.1) commutes and **(ii)** f is linear on fibers. Moreover, we require the map $\sigma \mapsto f \circ \sigma$ from $\Gamma(E)$ to $\mathrm{Vec}(M)$ to be injective.

$$
\begin{array}{ccc}
E & \xrightarrow{\;f\;} & TM \\
 & {\scriptstyle \pi_E}\searrow & \downarrow{\scriptstyle \pi} \\
 & & M
\end{array}
\tag{1.1}
$$

We say that a rank-varying distribution (E, f) is *orientable* if E is orientable as a vector bundle. Similarly, (E, f) is *trivializable* if E is isomorphic to the trivial bundle $M \times \mathbb{R}^k$.

Given a rank-varying distribution, we denote by $f_* : \Gamma(E) \to \mathrm{Vec}(M)$ the morphism of $\mathcal{C}^\infty(M)$-modules that maps $\sigma \in \Gamma(E)$ to $f \circ \sigma \in \mathrm{Vec}(M)$. The image under f_* of $\Gamma(E)$ is denoted by Δ. Given a point $q \in M$, the evaluation at q of elements in Δ is denoted by $\Delta(q)$ and coincides with the subspace $f(E_q)$, where $E_q = \pi_E^{-1}(q)$.

Let $\mathrm{Lie}(\Delta)$ be the smallest Lie subalgebra of $\mathrm{Vec}(M)$ containing Δ and $\mathrm{Lie}_q(\Delta) = \{V(q) \mid V \in \mathrm{Lie}(\Delta)\}$ for every $q \in M$. We say that (E, f) satisfies the *Lie bracket generating condition* if

$$\forall\, q \in M \quad \mathrm{Lie}_q(\Delta) = T_q M. \tag{1.2}$$

A property (P) defined for rank-varying distributions is said to be *generic* if for every vector bundle E of rank k over M, (P) holds for every f in an open and dense subset of the set of morphisms of vector bundles from E to TM inducing the identity on M, endowed with the $\mathcal{C}^\infty$-Whitney topology (see [42] for the definition of the $\mathcal{C}^\infty$-Whitney topology).

Remark 1.1. The Lie bracket generating condition (1.2) is a generic property for rank-varying distributions, whenever $k > 1$.

A generalized sub-Riemannian structure is defined by requiring that E is an Euclidean bundle.

Definition 1.2. A *rank-varying sub-Riemannian structure* is a triple $\mathcal{S} = (E, f, \langle \cdot, \cdot \rangle)$ where (E, f) is a rank-varying distribution satisfying the Lie bracket generating condition on a manifold M and $\langle \cdot, \cdot \rangle_q$ is a scalar product on E_q smoothly depending on q.

This definition generalizes several classical structures. First of all, a Riemannian manifold (M, g) is a rank-varying sub-Riemannian structure where $E = TM$, $f = 1_{TM}$ and $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$.

Classical sub-Riemannian structures (see [7, 33]) are rank-varying sub-Riemannian structures such that E is a proper Euclidean subbundle of TM and f is the inclusion.

Given k vector fields $f_1, \ldots, f_k \in \mathrm{Vec}(M)$, the driftless control-affine system

$$\dot{q} = \sum_{i=1}^{k} u_i f_i(q)$$

can be seen as the rank-varying sub-Riemannian structure $(E, f, \langle \cdot, \cdot \rangle)$ where E is the trivial bundle $M \times \mathbb{R}^k$, the morphism f is

$$f(q, (u_1, \ldots, u_k)) = \sum_{i=1}^{k} u_i f_i(q), \quad (q, (u_1, \ldots, u_k)) \in M \times \mathbb{R}^k,$$

and $\langle \cdot, \cdot \rangle$ is the canonical Euclidean structure on $\mathbb{R}^k$.

Finally, we introduce the subject of the following chapters.

Definition 1.3. An *n-dimensional almost-Riemannian structure* (n-ARS for short) is a rank-varying sub-Riemannian structure where $k = n$, i.e., the rank of the vector bundle E is equal to the dimension of the manifold M.

Let $\mathcal{S} = (E, f, \langle \cdot, \cdot \rangle)$ be a rank-varying sub-Riemannian structure. The Euclidean structure on E and the injectivity of the morphism f_* allow to define a symmetric positive definite $\mathcal{C}^\infty(M)$-bilinear form on the submodule Δ by

$$\begin{aligned} G : \Delta \times \Delta &\to \mathcal{C}^\infty(M) \\ (V, W) &\mapsto \langle \sigma_V, \sigma_W \rangle, \end{aligned}$$

where σ_V, σ_W are the unique sections of E such that $f \circ \sigma_V = V, f \circ \sigma_W = W$.

If $\sigma_1, \ldots, \sigma_k$ is an orthonormal frame for $\langle \cdot, \cdot \rangle$ on an open subset Ω of M, an *orthonormal frame for G* on Ω is given by $f \circ \sigma_1, \ldots, f \circ \sigma_k$. Orthonormal frames are systems of local generators of Δ.

For every $q \in M$ and every $v \in \Delta(q)$ define

$$\mathbf{G}_q(v) = \inf\{\langle u, u \rangle_q \mid u \in E_q, f(u) = v\}.$$

On one hand, G is a tensor at points q where $f|_{E_q}$ is an isomorphism; on the other hand, at points where f has not maximal rank we have the inequality

$$G(V, V)|_q \geq \mathbf{G}_q(V(q)).$$

Definition 1.4. A curve $\gamma : [0, T] \to M$ absolutely continuous with respect to the differential structure of M is said to be *admissible* for $\mathcal{S}$ if there exists a measurable essentially bounded function

$$[0, T] \ni t \mapsto u(t) \in E_{\gamma(t)}$$

called *control function*, such that $\dot{\gamma}(t) = f(u(t))$ for almost every $t \in [0, T]$.

Given an admissible curve $\gamma : [0, T] \to M$, the *length of γ* is

$$\ell(\gamma) = \int_0^T \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))}\, dt. \tag{1.3}$$

The distance induced by $\mathcal{S}$ on M is defined as

$$d(q_0, q_1) = \inf\{\ell(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ admissible}\}. \tag{1.4}$$

The finiteness and the continuity of $d(\cdot, \cdot)$ with respect to the topology of M are guaranteed by the Lie bracket generating assumption on the rank-varying sub-Riemannian structure.

Theorem 1.1 (Rashevsky-Chow). *Let $\mathcal{S}$ be a rank-varying sub-Riemannian structure on a connected manifold M. Then* **(i)** *$d(q, p) < +\infty$ for every $p, q \in M$ and* **(ii)** *$d : M \times M \to \mathbb{R}$ is continuous.*

Theorem 1.1 is a consequence of the Orbit Theorem (see [7, Theorem 5.1]) and implies that the distance function $d(\cdot, \cdot)$ endows M with the structure of a metric space compatible with the topology of M as a smooth manifold. The finiteness of the distance between two points implies in particular that there exist admissible curves connecting the two points. A natural question is wether some of these curves realize the distance between the two points. This leads us to the analysis of admissible curves that minimize the length functional $l(\cdot)$.

The functional $\ell(\gamma)$ is invariant under reparameterization of the admissible curve γ. Define the *energy functional*

$$J(\gamma) = \int_0^T \mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))\, dt.$$

If an admissible curve $\gamma : [0, T] \to M$ is a minimizer of $\ell(\cdot)$ on the set of admissible curves connecting q_0 to q_1 such that $v(t) = \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))}$ is constant, then γ minimizes the energy functional on the set of (admissible) curves $c : [0, \ell(\gamma)/v(0)] \to M$ connecting q_0 to q_1. Conversely, if an admissible curve $\gamma : [0, T] \to M$ minimizes the energy on the set of admissible curves connecting q_0 to q_1, then $\sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))}$ is constant and γ minimizes $\ell(\cdot)$ on the set of admissible curves connecting q_0 to q_1.

Definition 1.5. A *geodesic* is an admissible curve $\gamma : [0, T] \to M$ such that for every sufficiently small interval $[t_1, t_2] \subset [0, T]$, $\gamma|_{[t_1,t_2]}$ is a minimizer of $J(\cdot)$. A geodesic for which $\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))$ is (constantly) equal to one is said to be parameterized by *arclength*.

The existence of minimizers is provided by an important result, Filippov Theorem (see [7, Theorem 10.1]), whose consequence in the context of rank-varying sub-Riemannian structures is a generalization of the well-known fact in the Riemannian case that every point admits a neighborhood O such that every pair of points in O is connected by a geodesic. Once the existence of optimal curves is established, we are interested in computing them. A powerful tool to face this problem is the Pontryagin Maximum Principle (PMP) which provide first-order necessary conditions for an admissible curve to be optimal. We refer the reader to Chapter 12 in [7] for a complete treatement of this important result. As in this thesis we deal only with two-dimensional almost-Riemannian structures, we find it useful to state here a direct consequence of PMP in our specific context. In view of later applications in the thesis, we consider as initial and final conditions not only points, but submanifolds as well.

Let $\mathcal{S}$ be a 2-ARS on M and define on T^*M the Hamiltonian

$$H(q, p) = \frac{1}{2}(\langle p, X(q)\rangle^2 + \langle p, Y(q)\rangle^2), \qquad q \in \Omega, \;\; p \in T_q^*M, \tag{1.5}$$

where X, Y is a local orthonormal frame for G on Ω and $\langle \cdot, \cdot \rangle$ denotes the pairing between T^*M and TM. (Remark that the notation $\langle \cdot, \cdot \rangle$ for the Euclidean structure on the vector bundle E defining $\mathcal{S}$ and for the pairing between T^*M and TM is not ambiguous, the arguments being respectively two sections of E or a covector and a tangent vector to M). Notice that the function H is well-defined on the whole cotangent bundle and does not depend on the chosen local orthonormal frame of the vector bundle E.

Proposition 1.2. *Consider the minimization problem*

$$\int_0^T \mathbf{G}_{q(t)}(\dot{q}(t)) \, dt \to \min, \quad q(0) \in M_{\text{in}}, \quad q(T) \in M_{\text{fin}}, \tag{1.6}$$

where the minimum is taken over the set of admissible curves $q : [0, T] \to M$, M_{in} and M_{fin} are two submanifolds of M, and the final time $T > 0$ is fixed. Then every solution of (1.6) is the projection on M of a solution $(q(t), p(t))$ of the Hamiltonian system associated with H satisfying $p(0) \perp T_{q(0)}M_{\text{in}}$, $p(T) \perp T_{q(T)}M_{\text{fin}}$, and $H(q(t), p(t)) > 0$.

The simple form of the statement above is a consequence of the absence of abnormal minimizers, which follows from the Lie bracket generating assumption. As a consequence a curve is a geodesic if and only if it is the projection of a normal extremal.

Notice that H is constant along any given solution of the Hamiltonian system. Moreover, $H = 1/2$ if and only if $q(\cdot)$ is parameterized by arclength.

Given a subset W of M define $C^W = \{\lambda = (q,p) \in T^*M \mid q \in W, \ H(q,p) = 1/2, \ p \perp T_qW\}$. To simplify the notation, assume that all geodesics are defined in $[0, \infty[$. Define

$$\mathcal{E}_W : [0, \infty[\times C^W \to M$$
$$(t, \lambda) \mapsto \pi(e^{t\vec{H}}\lambda),$$

where $\pi : T^*M \to M$ is the canonical projection and $\vec{H}$ is the Hamiltonian vector field on T^*M associated with (1.5), i.e., $\mathcal{E}_W(t, \lambda)$ is the projection on M of the solution at time t of the Hamiltonian system with initial condition $\lambda(0) = \lambda$. When $W = \{q\}$, the map $\mathcal{E}_{\{q\}}$ is called *exponential map at q*. For every $\lambda \in C^W$, the *first conjugate time* is

$$t(\lambda) = \min\{t > 0, (t, \lambda) \text{ is a critical point of } \mathcal{E}_W\}.$$

and the *first conjugate locus from W* is $\{\mathcal{E}_W(t(\lambda), \lambda) \mid \lambda \in C^W\}$. The *cut locus $\mathbf{K}_W$ from W* is the set of points q' for which there exists a geodesic realizing the distance between W and q' which loses optimality after q'. It is well known (see for instance [1] for a proof in the three-dimensional contact case) that if $q' \in \mathbf{K}_W$ then one of the following two possibilities happen: **i)** more than one minimizing geodesics reach q'; **ii)** q' belongs to the first conjugate locus from W.

1.2 2-dimensional almost-Riemannian structures

The *singular set* (or singular locus) of an ARS on a two-dimensional manifold M is the set of points q where $\Delta(q) = f(E_q)$ has not maximal rank, that is,

$$\mathcal{Z} = \{q \in M \mid \dim(\Delta(q)) = 1\}.$$

Notice that $\dim(\Delta(q)) > 0$ for every point $q \in M$, because Δ is a bracket generating distribution.

Let us introduce the main hypothesis we will assume on 2-ARSs. We say that a 2-ARS *satisfies condition* (H0) if the following properties hold: **(i)** $\mathcal{Z}$ is an embedded one-dimensional submanifold of M; **(ii)** the points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; **(iii)** $\Delta_3(q) = T_qM$ for every $q \in M$, where $\Delta_1 = \Delta$ and $\Delta_{k+1} = \Delta_k + [\Delta, \Delta_k]$. Using a transversality argument, it is not hard to see that property (H0) is generic for 2-ARSs.

Definition 1.6. A *local representation* of a 2-ARS at a point $q \in M$ is a pair of vector fields (X, Y) on $\mathbb{R}^2$ such that there exist: a neighborhood U of q in M, a neighborhood V of $(0,0)$ in $\mathbb{R}^2$, a diffeomorphism $\varphi : U \to V$, and a local orthonormal frame (F_1, F_2) for the ARS around q such that $\varphi(q) = (0,0)$ and $\varphi_*F_1 = X$, $\varphi_*F_2 = Y$, where φ_* denotes the push-forward.

For simplicity, in what follows we omit the diffeomorphism φ. ARSs satisfying hypothesis (H0) admit the following local representations.

Theorem 1.3. *Given a 2-ARS $(E, f, \langle \cdot, \cdot \rangle)$ satisfying (H0), for every point $q \in M$ there exist a neighborhood U of q and an orthonormal frame (X, Y) for the ARS on U, such that up to a smooth change of coordinates defined on U, $q = (0,0)$ and (X, Y) has one of the forms*

$$
\begin{aligned}
\text{(F1)} \qquad & X(x, y) = (1, 0), \quad Y(x, y) = (0, e^{\phi(x,y)}), \\
\text{(F2)} \qquad & X(x, y) = (1, 0), \quad Y(x, y) = (0, x e^{\phi(x,y)}), \\
\text{(F3)} \qquad & X(x, y) = (1, 0), \quad Y(x, y) = (0, (y - x^2 \psi(x)) e^{\xi(x,y)}),
\end{aligned}
$$

where ϕ, ξ and ψ are smooth real-valued functions such that $\phi(0, y) = 0$ and $\psi(0) > 0$.

Theorem 1.3 appeared in [3, Theorem 1]. To our purpouses, we find it useful to recall here the proof given by the authors in the cited paper. First let us recall an auxiliary result (see [3, Lemma 1]).

Lemma 1.4. *Under the hypothesis of Theorem 1.3, let W be an embedded one-dimensional submanifold of M. Assume $\Delta(q) + T_q W = T_q M$ for every $q \in W$. Then, for every $q \in W$ there exist a neighborhood U, a local orthonormal frame (X, Y) for the ARS on U, and a coordinate system such that $q = (0,0)$ and **(i)** $W \cap U = \{(0, h) \mid h \in \mathbb{R}\}$; **(ii)** $X(x, y) = (1, 0)$ and $Y(x, y) = (0, a(x, y))$, where $a \in \mathcal{C}^\infty(U)$.*

Proof. Let $q \in W$ and $\mathbb{R} \ni \alpha \mapsto w(\alpha)$ be a smooth regular parametrization of W such that $q = w(0)$. Since W is transversal to the distribution, there exists a smooth choice $\alpha \mapsto p_0(\alpha) \in T^*_{w(\alpha)} M$ of a covector such that $p_0(\alpha) \perp T_{w(\alpha)} W$ and $H(w(\alpha), p_0(\alpha)) \equiv \frac{1}{2}$, where H is the Hamiltonian defined in (1.5). Denote by $\bar{\mathcal{E}}(t, \alpha)$ the projection on M of the solution at time t of the Hamiltonian system defined by H on $T^* M$ with initial condition $(q(0), p(0)) = (w(\alpha), p_0(\alpha))$. Let us show that the two vectors

$$
v_1 = \frac{\partial \bar{\mathcal{E}}}{\partial \alpha}(0, 0) \quad \text{and} \quad v_2 = \frac{\partial \bar{\mathcal{E}}}{\partial t}(0, 0)
$$

are linearly independent. By construction, $v_1 = \frac{dw}{d\alpha}(0) \in T_q W$ whence $\langle p_0(0), v_1 \rangle = 0$. Recalling that H is quadratic in the covector and it is normalized to $1/2$, we have

$$
\langle p_0(0), v_2 \rangle = \left\langle p_0(0), \frac{\partial H}{\partial \lambda}(q, p_0(0)) \right\rangle = 2H(q, p_0(0)) = 1.
$$

Thus v_2 is transversal to $T_q W$. This implies that $\bar{\mathcal{E}}$ is a local diffeomorphism at $(0, 0)$, i.e., there exists a neighborhood V of $(0, 0)$ and a neighborhood U of q such that $\bar{\mathcal{E}} : V \to U$ is a diffeomorphism. Consider the coordinate system (t, α) on U and define the vector field X by

$$
X(t, \alpha) = \frac{\partial \bar{\mathcal{E}}(t, \alpha)}{\partial t}.
$$

Since $\bar{\mathcal{E}}$ is the projection of a solution of the Hamiltonian system defined by the ARS, we have that $X \in \Delta$, i.e., $X = f \circ \sigma$, where $\sigma \in \Gamma(U)$. Moreover, since the solution $\bar{\mathcal{E}}(t, \alpha)$ is parametrized by arclength, we have that for every $p \in U$

$$\langle \sigma, \sigma \rangle \,|_p = G(X, X)|_p = \mathbf{G}_p(X(p)) = 1. \tag{1.7}$$

Let $\rho \in \Gamma(U)$ be such that $\{\sigma, \rho\}$ is a local orthonormal frame for $\langle \cdot, \cdot \rangle$, and define $Y = f \circ \rho$.

The statement is proved if we show that the first component of Y in coordinates (t, α) is identically equal to zero. To this aim, let $q' \in U \cap \mathcal{Z}$. Thanks to (1.7), $f(\rho(q')) = 0$ and by construction $d\alpha(Y)|_{q'} = 0$, whence $dt(Y)|_{q'} = 0$. Let now $q' \in U \setminus \mathcal{Z}$, $q' = (\bar{t}, \bar{\alpha})$. Then imposing that the curve $\gamma(s) = (s + \bar{t}, \alpha)$ is the geodesic for the ARS minimizing the distance between q' and the α-axis we easily find $dt(Y)|_{q'} = 0$. ∎

Proof of Theorem 1.3. Let $q \in M$. Thanks to the third assumption in hypothesis (H0) there are three possibilities: $\Delta(q) = T_q M$, $\Delta(q) \subsetneq \Delta_2(q) = T_q M$, or $\Delta_2(q) \subsetneq \Delta_3(q) = T_q M$.

Assume $\Delta(q) = T_q M$. Let W be any embedded (smooth) curve passing through q. Applying Lemma 1.4 we build a coordinate system (x, y) on a neighborhood U of q such that an orthonormal frame is given by $X(x, y) = (1, 0)$, $Y(x, y) = (0, a(x, y))$. Since $\mathcal{Z}$ is one-dimensional and $q \notin \mathcal{Z}$, we may assume $\Delta(p) = T_p M$ for every $p \in U$, whence $a(x, y) \neq 0$ on U. Applying the coordinate change $x' = x$, $y' = \int_0^y 1/a(0, s) ds$ we get the local representation (F1) with $\phi(x, y) = \log(a(x, y)/a(0, y))$.

Assume $\Delta(q) \subsetneq T_q M$, i.e., $q \in \mathcal{Z}$. Let V_1, V_2 be any local orthonormal frame around q and $V_1(q) \neq 0$. Denote by $(V_1 V_2)$ the matrix whose columns are the components of the vector fields V_1, V_2 in a chosen coordinate system. By definition, the singular set $\mathcal{Z}$ in a neighborhood of q coincide with the set $\{p \mid \det(V_1 V_2)|_p = 0\}$ and

$$L_{V_1}(\det(V_1, V_2)) = \det(V_1, [V_1, V_2]) + \operatorname{div} V_1 \det(V_1 V_2),$$

where L_V denotes the Lie derivative along a vector field V and div V is the divergence of V. Evaluating the last identity at the point $q \in \mathcal{Z}$, it follows that $\Delta(q)$ is transversal to $T_q \mathcal{Z}$ if and only if $\Delta_2(q) = T_q M$ and $\Delta(q)$ is tangent to $\mathcal{Z}$ if and only if $\Delta_2(q) \subsetneq T_q M$.

Let $\Delta_2(q) = T_q M$. Then, by condition **(ii)** in hypothesis (H0), $\Delta(p)$ is transversal to $T_p \mathcal{Z}$ for all points in a neighborhood of q. Hence we can apply Lemma 1.4 with $W = \mathcal{Z}$ to get a local coordinate system (x, y) in a neighborhood U of q and a local orthonormal frame of the type $X(x, y) = (1, 0)$, $Y(x, y) = (0, a(x, y))$. In these coordinates $\mathcal{Z} \cap U = \{(x, y) \mid x = 0\}$ and $a(0, y) \equiv 0$, X and Y being linearly dependent on $\mathcal{Z}$. Moreover, since $\Delta_2(q) = T_q M$, we have that $[X, Y]|_{(0,y)} = (0, \partial_x a(0, y))$ is transversal to $X(0, y)$, i.e., $\partial_x a(0, y) \neq 0$. Hence, possibly reducing U, we assume $a(x, y) = x \bar{a}(x, y)$, with $\bar{a}(x, y) \neq 0$ for all $(x, y) \in U$. Considering the coordinate change $x' = x$, $y' = \int_0^y 1/\bar{a}(0, s) ds$ we find the required form for the orthonormal frame with $\phi(x, y) = \log(\bar{a}(x, y)/\bar{a}(0, y))$.

Finally, consider the case $\Delta_2(q) \subsetneq \Delta_3(q) = T_q M$. Take any smooth curve W passing through q transversally to $\Delta(q)$ and apply Lemma 1.4. Then we get a

coordinate system on a neighborhood U of q and an orthonormal frame of the type $X(x, y) = (1, 0)$, $Y(x, y) = (0, a(x, y))$. Since $\mathcal{Z}$ is an embedded submanifold of M and $T_q\mathcal{Z} = \mathrm{span}\{X(0, 0)\}$, there exists $\zeta \in \mathcal{C}^\infty(\mathbb{R})$ such that, possibly reducing U, $\mathcal{Z} \cap U = \{(x, y) \mid y = \zeta(x)\}$. Hence we have $a(x, y) = (y - \zeta(x))\overline{a}(x, y)$. Recalling that $\Delta(q) = \Delta_2(q)$, we have $a(0, 0) = \partial_x a(0, 0) = 0$ and $\partial_x^2 a(0, 0) \neq 0$. Thus $\zeta''(0) \neq 0$ and $\overline{a}(0, 0) \neq 0$. Hence, $\zeta(x) = x^2\tilde{\psi}(x)$ with $\tilde{\psi}(0) \neq 0$ and, eventually replacing Y by $-Y$, we can assume $\overline{a}(0, 0) > 0$. Finally, taking the coordinate change $x' = x$, $y' = \mathrm{sign}(\tilde{\psi}(0))y$ and possibly reducing U, we end up with the local representation (F3) where $\xi(x, y) = \log(\overline{a}(x, y))$ and $\psi(0) > 0$.

∎

Remark 1.2. Notice that the construction of local representations in Theorem 1.3 depends on the chosen curve W as well as on the chosen parametrization of W.

Definition 1.7. Let $\mathcal{S}$ be a 2-ARS satisfying (H0). A point $q \in M$ is said to be an *ordinary point* if $\Delta(q) = T_qM$, hence, if $\mathcal{S}$ is locally described by (F1). We call q a *Grushin point* if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_qM$, i.e. if the local description (F2) applies. Finally, if $\Delta(q) = \Delta_2(q)$ has dimension one and $\Delta_3(q) = T_qM$ then we say that q is a *tangency point* and $\mathcal{S}$ can be described near q by the local representation (F3). We define

$$\mathcal{T} = \{q \in \mathcal{Z} \mid q \text{ tangency point of } \mathcal{S}\}.$$

Grushin points, respectively tangency points, are characterized by the distribution being transversal, respectively tangent, to the singular set.

Global results on almost-Riemannian surfaces

We focus our study on a special class of rank-varying sub-Riemannian structures, namely, 2-dimensional almost-Riemannian structures (2-ARS for short). In this chapter we investigate topological and metric aspects of 2-ARSs from a global viewpoint. The first important result (Theorem 2.1) is a classification of generic oriented 2-ARS on compact oriented surfaces by means of the Euler number of the vector bundle associated with it. Notice that the Euler number identifies a vector bundle in the class of oriented rank-2 vector bundles over a compact oriented surfaces and it measures how far the vector bundle is from the trivial one. The theorem essentially relates the Euler number of the vector bundle defining the structure with the topology of M and the contributions due to tangency points. As a direct consequence we find a necessary and sufficient condition for a 2-ARS to be trivializable. Moreover, we show a Gauss–Bonnet formula for generic oriented 2-ARS with tangency points on compact oriented surfaces (Theorem 2.6). To this aim, the first problem is to give a notion of integrability of the curvature. Starting from the definition of integrability chosen in [3] when tangency points are absent, we generalize it taking into account that the distance from the singular set has different orders as a point approaches the tangency point from the two sides of $\mathcal{Z}$. Then our result consists of showing the (suitably defined) integral of the curvature exists and is equal to $2\pi e(E)$, where $e(E)$ denotes the Euler number of the vector bundle associated with the structure. Such result generalizes the Gauss–Bonnet formula given in [3] for structures having no tangency point.

The last part of the chapter is devoted to the analysis of 2-ARS from the metric point of view and, in particular, to the problem of Lipschitz equivalence among different almost-Riemannian distances. In the Riemannian case, the Lipschitz equivalence classification coincides with the differential equivalence of the underlying surfaces. On the contrary, such classification is finer for almost-Riemannian distances. We show that the Lipschitz equivalence class of a 2-ARS is determined by how the singular set splits the surface and by the location of tangency points (with their contributions). All the information needed to identify the Lipschitz equivalence class of an ARS can be encoded in a labelled graph that is naturally associated with the structure. The main result (Theorem 2.9) provides the characterization of the Lipschitz equivalence class of an almost-Riemannian distance in terms of the labelled graph associated with it.

The chapter is organized as follows. Section 2.1.1 introduces the definition of the

number of revolutions of a 2-ARS. In Section 2.1.2 we state and prove Theorem 2.1. First we construct a section of the sphere bundle on a tubular neighborhood of the singular locus having each tangency point as singularity. Then, we extend the section to the complement in the manifold of a finite set and compute the sum of the indices at all the singularities. In Section 2.2, we first recall the Gauss–Bonnet formula for 2-ARS without tangency points proved in [3]. Then, the relation between tangency points and integrability of the curvature with respect to the area form associated with an almost-Riemannian structure is discussed. In particular, we provide in Section 2.2.1 some numerical simulations strongly hinting that in presence of tangency points the integral of the curvature defined in [3] does not converge. This leads us to introduce in Section 2.2.2 the notion of 3-scale $\mathcal{S}$-integrability. Thanks to a Gauss–Bonnet formula for almost-Riemannian surfaces with boundary given in [17], we compute in Section 2.2.3 the total curvature of a generic 2-ARS with tangency points, proving Theorem 2.6. In Section 2.3 we consider the problem of Lipschitz equivalence of almost-Riemannian distances on surfaces. We start by defining in Section 2.3.1 the graph associated with an almost-Riemannian structure. Then, in Section 2.3.2 we state and give the proof of Theorem 2.9. We show that having equivalent graphs is a necessary condition for Lipschitz equivalent structures. Finally, we prove this condition to be sufficient.

Unless specified, the results given in sections 2.1, 2.2 are to be found in [6] and the results in section 2.3 in [16].

2.1 Topological classification of 2-ARS

2.1.1 Number of revolutions of Δ

Let M be a compact surface and let $\mathcal{S} = (E, f, \langle\cdot, \cdot\rangle)$ be an ARS on M satisfying (H0). Assume $\mathcal{S}$ and M to be oriented. Thanks to the hypothesis (H0), $M \setminus \mathcal{Z}$ splits into two open sets M^+ and M^- such that $f : E|_{M^+} \to TM^+$ is an orientation-preserving isomorphism and $f : E|_{M^-} \to TM^-$ is an orientation-reversing isomorphism.

Fix on $\mathcal{Z}$ the orientation induced by M^+ and consider a connected component W of $\mathcal{Z}$. Let $V \in \Gamma(TW)$ be a never-vanishing vector field whose duality product with the fixed orientation on W is positive. Since M is oriented, $TM|_W$ is isomorphic to the trivial bundle of rank two over W. We choose an isomorphism $t : TM|_W \to W \times \mathbb{R}^2$ such that t is orientation-preserving and for every $q \in W$, $t \circ V(q) = (q, (1, 0))$. This trivialization induces an orientation-preserving isomorphism between the projectivization of $TM|_W$ and $W \times S^1$. For the sake of readability, in what follows we omit the isomorphism t and identify $TM|_W$ (respectively, its projectivization) with $W \times \mathbb{R}^2$ (respectively, $W \times S^1$).

Since $\Delta|_W$ is a subbundle of rank one of $TM|_W$, $\Delta|_W$ can be seen as a section of the projectivization of $TM|_W$, i.e., a smooth map (still denoted by Δ) $\Delta : W \to W \times S^1$ such that $\pi_1 \circ \Delta = \mathrm{Id}_W$, where $\pi_1 : W \times S^1 \to W$ denotes the projection on the first component. We define $\tau(\Delta, W)$, the *number of revolutions* of Δ along W, to be the degree of the map $\pi_2 \circ \Delta : W \to S^1$, where $\pi_2 : W \times S^1 \to S^1$ is the projection on the second component. Notice that $\tau(\Delta, W)$ changes sign if we reverse the orientation of W.

Let us show how to compute $\tau(\Delta, W)$. By construction, $\pi_2 \circ V : W \to S^1$ is constant. Let $\pi_2 \circ V(q) \equiv \theta_0$. Since $\Delta_3(q) = T_q M$ for every $q \in M$, θ_0 is a regular value of $\pi_2 \circ \Delta$. By definition,

$$\tau(\Delta, W) = \sum_{q|\pi_2 \circ \Delta(q) = \theta_0} \text{sign}(d_q(\pi_2 \circ \Delta)) = \sum_{q \in W \cap \mathcal{T}} \text{sign}(d_q(\pi_2 \circ \Delta)), \qquad (2.1)$$

where $\mathcal{T}$ is the set of tangency points, d_q denotes the differential at q of a smooth map, and $\text{sign}(d_q(\pi_2 \circ \Delta)) = 1$, respectively -1, if $d_q(\pi_2 \circ \Delta)$ preserves, respectively reverses, the orientation. The last equality in (2.1) follows from the fact that a point q satisfies $\pi_2 \circ \Delta(q) = \theta_0$ if and only if $\Delta(q)$ is tangent to W at q, i.e., $q \in \mathcal{T}$.

Define the *contribution at a tangency point* q as $\tau_q = \text{sign}(d_q(\pi_2 \circ \Delta))$ (see Figure 2.1). Moreover, we define

$$\tau(\mathcal{S}) = \sum_{W \in \mathfrak{C}(\mathcal{Z})} \tau(\Delta, W),$$

where $\mathfrak{C}(\mathcal{Z}) = \{W \mid W \text{ connected component of } \mathcal{Z}\}$. Clearly, $\tau(\mathcal{S}) = \sum_{q \in \mathcal{T}} \tau_q$.
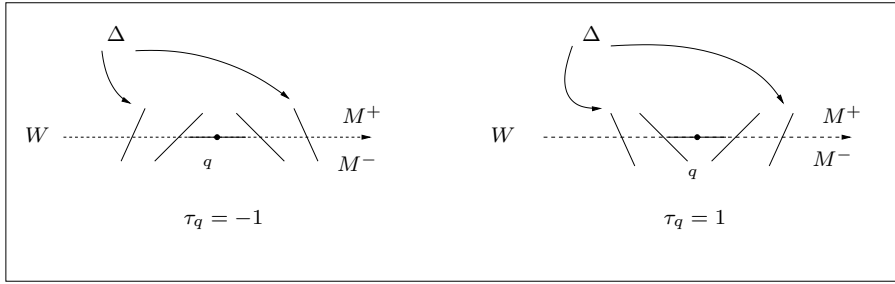
Figure 2.1: Tangency points with opposite contributions

2.1.2 The main result

To classify 2-ARSs we recall the notion of Euler number of a vector bundle. Given an oriented vector bundle of rank n over a compact connected oriented n-manifold M, the Euler number of E, denoted by $e(E)$, is the self-intersection number of M in E, where M is identified with the zero section. To compute $e(E)$, consider a smooth section $\sigma : M \to E$ transverse to the zero section. Then, by definition,

$$e(E) = \sum_{p|\sigma(p)=0} i(p, \sigma),$$

where $i(p, \sigma) = 1$, respectively -1, if $d_p\sigma : T_p M \to T_{\sigma(p)} E$ preserves, respectively reverses, the orientation. Notice that if we reverse the orientation on M or on E then $e(E)$ changes sign. Hence, the Euler number of an orientable vector bundle E is defined up to a sign, depending on the orientations of both E and M. Since reversing the orientation on M also reverses the orientation of TM, the Euler number of TM is defined unambiguously and is equal to $\chi(M)$, the Euler characteristic of M. We refer the reader to [29] for a more detailed discussion of this subject.

Remark 2.1. Assume that $\sigma \in \Gamma(E)$ has only isolated zeros, i.e., the set $\{p \mid \sigma(p) = 0\}$ is finite. If E is endowed with a smooth scalar product $\langle \cdot, \cdot \rangle$, we define $\tilde{\sigma} : M \setminus \{p \mid \sigma(p) = 0\} \to SE$ by $\tilde{\sigma}(q) = \frac{\sigma(q)}{\sqrt{\langle \sigma(q), \sigma(q) \rangle}}$, where SE is the sphere bundle associated with E. Then if $\sigma(p) = 0$, $i(p, \tilde{\sigma}) = i(p, \sigma)$ is equal to the degree of the map $\partial B \to S^{n-1}$ that associate with each $q \in \partial B$ the value $\tilde{\sigma}(q)$, where B is a neighborhood of p diffeomorphic to an open ball in $\mathbb{R}^n$ that does not contain any other zero of σ.

Notice that if $i(p, \sigma) \neq 0$, the limit $\lim_{q \to p} \tilde{\sigma}(q)$ does not exist and, in this case, we say that $\tilde{\sigma}$ has a singularity at p.

The following result classifies almost-Riemannian structures using the Euler number of the vector bundle associated with it. Recall that a 2-ARS is oriented if the vector bundle E is oriented.

Theorem 2.1. *Let M be a compact oriented surface endowed with an oriented almost-Riemannian structure $\mathcal{S} = (E, f, \langle \cdot, \cdot \rangle)$ satisfying the generic hypothesis (H0). Then $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = e(E)$.*

Remark 2.2. Notice that the Euler number $e(E)$ measures how far the vector bundle E is from the trivial one. Indeed, if E is an oriented rank-2 vector bundle over a compact oriented surface, then E is isomorphic to the trivial bundle if and only if $e(E) = 0$.

A direct consequence of Theorem 2.1 is the following.

Corollary 2.2. *Under the assumptions of Theorem 2.1, the structure $\mathcal{S}$ is trivializable if and only if $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$.*

Proof of Theorem 2.1

The idea of the proof is to find a section σ of SE with isolated singularities $p_1, \ldots, p_m$ such that $\sum_{j=1}^{m} i(p_j, \sigma) = \chi(M^+) - \chi(M^-) + \tau(\mathcal{S})$. In the sequel, we consider $\mathcal{Z}$ to be oriented with the orientation induced by M^+.

We start by defining σ on a neighborhood of $\mathcal{Z}$. Let W be a connected component of $\mathcal{Z}$. Since M is oriented, there exists an open tubular neighborhood $\mathbf{W}$ of W and a diffeomorphism $\Psi : S^1 \times (-1, 1) \to \mathbf{W}$ that preserves the orientation and $\Psi|_{S^1 \times \{0\}}$ is an orientation-preserving diffeomorphism between S^1 and W. Remark that $f : E|_{\mathbf{W}^+} \to T\mathbf{W}^+$ is an orientation-preserving isomorphism of vector bundles, while $f : E|_{\mathbf{W}^-} \to T\mathbf{W}^-$ is an orientation-reversing isomorphism of vector bundles, where $\mathbf{W}^{\pm} = \mathbf{W} \cap M^{\pm}$. For every $s \neq 0$, lift the tangent vector to $\theta \mapsto \Psi(\theta, s)$ to E using f^{-1}, rotate it by the angle $\pi/2$ and normalize it: σ is defined as this unit vector (belonging to $E_{\Psi(\theta,s)}$) if $s > 0$, its opposite if $s < 0$. In other words, $\sigma : \mathbf{W} \setminus W \to SE$ is given by

$$\sigma(q) = \text{sign}(s) \frac{R_{\pi/2} f^{-1}(\frac{\partial \Psi}{\partial \theta}(\theta, s))}{\sqrt{\langle f^{-1}(\frac{\partial \Psi}{\partial \theta}(\theta, s)), f^{-1}(\frac{\partial \Psi}{\partial \theta}(\theta, s)) \rangle}}, \quad (\theta, s) = \Psi^{-1}(q), \qquad (2.2)$$

where $R_{\pi/2}$ denotes the rotation (with respect to the Euclidean structure) in E by angle $\pi/2$ in the counterclockwise sense. The following lemma shows that σ can be extended to a continuous section from $\mathbf{W} \setminus \mathcal{T}$ to SE.

Lemma 2.3. *σ can be continuously extended to every point $q \in W \setminus \mathcal{T}$.*

Proof. Let $q \in W \setminus \mathcal{T}$, U be a neighborhood of q in M and (x,y) be a system of coordinates on U centered at q such the almost-Riemannian structure has the form (F2) (see Theorem 1.3). Assume, moreover, that U is a trivializing neighborhood of both E and TM and the pair of vector fields (X,Y) is the image under f of a positively-oriented local orthonormal frame of E. Then $W \cap U = \{(x,y) \mid x = 0\}$. Since $\frac{\partial \Psi}{\partial \theta}(\theta, 0)$ is non-zero and tangent to W, $\frac{\partial \Psi}{\partial \theta}(\theta, 0)$ is tangent to the y-axis. Hence, thanks to the Preparation Theorem [32], there exist $h_2 : \mathbb{R} \to \mathbb{R}$, $h_1, h_3 : \mathbb{R}^2 \to \mathbb{R}$ smooth functions such that $h_2(y) \neq 0$ for every $y \in \mathbb{R}$ and for $\Psi(\theta, s) \in U$

$$\frac{\partial \Psi}{\partial \theta}(\theta, s) = (xh_1(x,y), h_2(y) + xh_3(x,y)),$$

where (x,y) are the coordinates of the point $\Psi(\theta, s)$. Let us compute σ at a point $p \in (\mathbf{W} \cap U) \setminus W$. Since

$$\frac{\partial \Psi}{\partial \theta}(\theta, s) = xh_1(x,y)X(x,y) + \frac{h_2(y) + xh_3(x,y)}{xe^{\phi(x,y)}}Y(x,y),$$

then

$$f^{-1}\left(\frac{\partial \Psi}{\partial \theta}(\theta, s)\right) = xh_1(x,y)\zeta(x,y) + \frac{h_2(y) + xh_3(x,y)}{xe^{\phi(x,y)}}\rho(x,y),$$

where (ζ, ρ) is the unique local orthonormal basis of $E|_U$ such that $f \circ \zeta = X$ and $f \circ \rho = Y$. Notice that $U \cap M^+ = \{(x,y) \mid x > 0\}$ and $U \cap M^- = \{(x,y) \mid x < 0\}$. Using formula (2.2), for $(x,y) = \Psi(\theta, s) \in U \setminus W$ one easily gets

$$\sigma(x,y) = \frac{\text{sign}(x)}{l(x,y)}\left(-\frac{h_2(y) + xh_3(x,y)}{xe^{\phi(x,y)}}\zeta + xh_1(x,y)\rho\right),$$

where

$$l(x,y) = \sqrt{x^2 h_1(x,y)^2 + \frac{(h_2(y) + xh_3(x,y))^2}{x^2 e^{2\phi(x,y)}}}.$$

Since

$$\lim_{x \to 0} \frac{\text{sign}(x)(h_2(y) + xh_3(x,y))}{l(x,y)xe^{\phi(x,y)}} = \frac{h_2(y)}{|h_2(y)|} \quad \text{and} \quad \lim_{x \to 0} \frac{\text{sign}(x)xh_1(x,y)}{l(x,y)} = 0,$$

σ can be continuously extended to the set $\{x = 0\} = W \cap U$. ∎

The next step of the proof is to show that for every $q \in W \cap \mathcal{T}$, $i(q, \sigma) = \tau_q$.

Lemma 2.4. *Let $\sigma : \mathbf{W} \setminus \mathcal{T} \to SE$ be the continuous section obtained in Lemma 2.3. Then, for every $q \in W \cap \mathcal{T}$ the index of σ at q is equal to τ_q and, consequently,*

$$\sum_{q \in \mathcal{T} \cap W} i(q, \sigma) = \tau(\Delta, W). \tag{2.3}$$

Proof. Let $q \in W \cap \mathcal{T}$, U be a neighborhood of q in M and (x,y) be a system of coordinates on U centered at q such the almost-Riemannian structure has the form (F3) (see Theorem 1.3) i.e. a local orthonormal frame (X,Y) is given by

$$X = (1,0), \qquad Y = (0, (y - x^2\psi(x))e^{\xi(x,y)}).$$

Define $\alpha = 1$, respectively -1, if (X, Y) is the image under f of a positively-oriented, respectively negatively-oriented, local orthonormal frame of E. One can check that $\tau_q = -\alpha$. Let us make the following change of coordinates

$$\tilde{x} = x, \qquad \tilde{y} = \alpha(y - x^2\psi(x)).$$

In these new coordinates, X and Y become

$$X = (1, -\alpha(2\tilde{x}\psi(\tilde{x}) + \tilde{x}^2\psi'(\tilde{x}))), \qquad Y = (0, \tilde{y}e^{\xi(\tilde{x},\alpha\tilde{y}+\tilde{x}^2\psi(\tilde{x}))})$$

and $W \cap U$ is the $\tilde{x}$-axis. In the following, to simplify notations, we omit the tildes and denote the function $\xi(\tilde{x}, \alpha\tilde{y} + \tilde{x}^2\psi(\tilde{x}))$ by $\xi(x, y)$. Since $\frac{\partial\psi}{\partial\theta}(\theta, 0)$ is tangent to W, by the Preparation Theorem [32] there exist $h_1 : \mathbb{R} \to \mathbb{R}$, $h_2, h_3 : \mathbb{R}^2 \to \mathbb{R}$ smooth functions such that $h_1(x) \neq 0$ for every $x \in \mathbb{R}$ and for $\Psi(\theta, s) \in U$

$$\frac{\partial\psi}{\partial\theta}(\theta, s) = (h_1(x) + yh_2(x, y), yh_3(x, y))$$

where (x, y) are the coordinates of the point $\Psi(\theta, s)$. This implies that

$$\frac{\partial\psi}{\partial\theta}(\theta, s) = (h_1(x) + yh_2(x, y))X(x, y) + $$
$$+ \frac{yh_3(x, y) + \alpha(h_1(x) + yh_2(x, y))(2x\psi(x) + x^2\psi'(x))}{ye^{\xi(x,y)}}Y(x, y).$$

Let (ζ, ρ) be the local orthonormal frame of E such that $X = f \circ \zeta$ and $Y = f \circ \rho$. From equation (2.2), it follows that

$$\sigma(x, y) = -\alpha\frac{\text{sign}(y)}{l(x, y)}\frac{yh_3(x, y) + \alpha(h_1(x) + yh_2(x, y))(2x\psi(x) + x^2\psi'(x))}{ye^{\xi(x,y)}}\zeta + $$
$$+ \alpha\frac{\text{sign}(y)}{l(x, y)}(h_1(x) + yh_2(x, y))\rho,$$

where

$$l(x, y) = \sqrt{(h_1(x) + yh_2(x, y))^2 + \left(\frac{yh_3(x, y) + \alpha(h_1(x) + yh_2(x, y))(2x\psi(x) + x^2\psi'(x))}{ye^{\xi(x,y)}}\right)^2}.$$

Notice that for $x = 0, y \neq 0$ we have

$$\sigma(0, y) = \frac{\alpha\,\text{sign}(y)}{\sqrt{(h_1(0) + yh_2(0, y))^2 + h_3(0, y)^2e^{-2\xi(0,y)}}}(-e^{-\xi(0,y)}h_3(0, y)\zeta + (h_1(0) + yh_2(0, y))\rho),$$

whence the limit of σ as (x, y) tends to $(0, 0)$ does not exist. Let us compute the index of σ at $q = (0, 0)$. Using Taylor expansions of the components of σ in the basis (ζ, ρ) we find

$$\sigma(x, y) = \alpha\frac{\text{sign}(y)}{l(x, y)}\left(-\left(\frac{yh_3(0, 0) + 2\alpha xh_1(0)\psi(0)}{ye^{\xi(0,0)}} + O(\sqrt{x^2 + y^2})\right)\zeta + (h_1(0) + O(\sqrt{x^2 + y^2}))\rho\right).$$
$$(2.4)$$

Take a circle $t \mapsto (r \cos t, r \sin t)$ of radius r centered at $(0,0)$ and assume r so small that $(0,0)$ is the unique singularity of σ on the closed disk of radius r. By definition, $i((0,0), \sigma)$ is half the degree of the map from the circle S^r to $\mathbb{R}/\pi\mathbb{Z}$ that associates to each point the angle between $\mathrm{span}(\sigma)$ and the ζ. Using (2.4), this angle is

$$a(x, y) = -\arctan\left(\frac{yh_1(0)e^{\xi(0,0)}}{yh_3(0,0) + 2\alpha x\psi(0)h_1(0)} + O(\sqrt{x^2 + y^2})\right).$$

Computing a along the curve $x(t) = r \cos t$, $y(t) = r \sin t$, we find

$$a(r \cos t, r \sin t) = -\arctan\left(\frac{\sin(t)h_1(0)e^{\xi(0,0)}}{\sin(t)h_3(0,0) + 2\alpha \cos(t)\psi(0)h_1(0)} + O(r)\right).$$

Hence, by letting r go to zero, we are left to compute the degree of the map $\tilde{a} : [0, 2\pi) \to [0, \pi)$ where

$$\tilde{a}(t) = -\arctan\left(\frac{\sin(t)h_1(0)e^{\xi(0,0)}}{\sin(t)h_3(0,0) + 2\alpha \cos(t)\psi(0)h_1(0)}\right).$$

Since zero is a regular value of $\tilde{a}$, the degree of $\tilde{a}$ is

$$\sum_{t \in [0,2\pi) \mid \tilde{a}(t)=0} \mathrm{sign}(\tilde{a}'(t)) = \mathrm{sign}(\tilde{a}'(0)) + \mathrm{sign}(\tilde{a}'(\pi)) = -2\alpha,$$

where the last equality follows from $\tilde{a}'(0) = \tilde{a}'(\pi) = -\alpha e^{\xi(0,0)}/(2\psi(0))$. Hence, $i(q, \sigma) = -\alpha$. Since $\tau_q = -\alpha$, the lemma is proved. ∎

Let $\tilde{\mathcal{Z}} = \coprod_{W \in \mathfrak{C}(\mathcal{Z})} S^1$ and consider an orientation-preserving diffeomorphism $\Psi : \tilde{\mathcal{Z}} \times (-1, 1) \to \coprod_{W \in \mathfrak{C}(\mathcal{Z})} \mathbf{W}$ such that $\Psi|_{\tilde{\mathcal{Z}} \times \{0\}}$ is an orientation-preserving diffeomorphism onto $\mathcal{Z}$. Applying Lemma 2.3 to every $W \in \mathfrak{C}(\mathcal{Z})$ and reducing, if necessary, the cylinders $\mathbf{W}$, we can assume that the set of singularities of σ on $\mathbf{U} = \coprod_{W \in \mathfrak{C}(\mathcal{Z})} \mathbf{W}$ is $\mathcal{T}$. Then $\sigma : \mathbf{U} \setminus \mathcal{T} \to SE$ is continuous. Moreover, by equation (2.3),

$$\sum_{q \in \mathcal{T}} i(q, \sigma) = \tau(\mathcal{S}).$$

Extend σ to $M \setminus \mathbf{U}$. By a transversality argument, we can assume that the extended section has only isolated singularities $\{p_1, \ldots, p_k\} \in M \setminus \mathcal{Z}$. Since

$$e(E) = \sum_{j=1}^{k} i(p_j, \sigma) + \sum_{q \in \mathcal{T}} i(q, \sigma) = \sum_{j=1}^{k} i(p_j, \sigma) + \tau(\mathcal{S}),$$

we are left to prove that

$$\sum_{j=1}^{k} i(p_j, \sigma) = \chi(M^+) - \chi(M^-). \tag{2.5}$$

To this aim, consider the vector field $F = f \circ \sigma$. F satisfies $G(F, F) \equiv 1$, where $G(\cdot, \cdot)$ is defined as in Chapter 1 and the set of singularities of $F|_{M \setminus \mathcal{Z}}$ is exactly

$\{p_1, \ldots, p_k\}$. Let us compute the index of F at a singularity $p \in \{p_1, \ldots, p_k\}$. Since $f : E|_{M^+} \to TM^+$ preserves the orientation and $f : E|_{M^-} \to TM^-$ reverses the orientation, it follows that $i(p, F) = \pm i(p, \sigma)$, if $p \in M^{\pm}$. Therefore,

$$\sum_{j=1}^{k} i(p_j, \sigma) = \sum_{j|p_j \in M^+} i(p_j, F) - \sum_{j|p_j \in M^-} i(p_j, F). \tag{2.6}$$

The theorem is proved if we show that

$$\sum_{j|p_j \in M^+} i(p_j, F) = \chi(M^+), \quad \sum_{j|p_j \in M^-} i(p_j, F) = \chi(M^-). \tag{2.7}$$

To deduce equation (2.7), define $N^+ = M^+ \setminus \Psi(\tilde{\mathcal{Z}} \times (0, 1/2))$. Notice that, by construction, $\sigma|_{\Psi(\tilde{\mathcal{Z}} \times \{1/2\})}$ is non-singular, hence the same is true for $F|_{\Psi(\tilde{\mathcal{Z}} \times \{1/2\})}$. Moreover, the almost-Riemannian angle between $T_q(\partial N^+)$ and $\mathrm{span}(F(q))$ is constantly equal to $\pi/2$. Hence $F|_{\partial N^+}$ points towards N^+ and applying the Hopf's Index Formula to every connected component of N^+ we conclude that

$$\sum_{j|p_j \in M^+} i(p_j, F) = \sum_{j|p_j \in N^+} i(p_j, F) = \chi(N^+) = \chi(M^+).$$

Similarly, we find

$$\sum_{j|p_j \in M^-} i(p_j, F) = \chi(M^-).$$

<div align="right">∎</div>

2.2   A Gauss–Bonnet formula for 2-ARSs

Let M be a compact oriented surface and let $\mathcal{S} = (E, f, \langle \cdot, \cdot \rangle)$ be an oriented 2-ARS on M. Notice that $\langle \cdot, \cdot \rangle$ defines a Riemannian structure on $M \setminus \mathcal{Z}$. Denote by K the Gaussian curvature of such a structure and by ω a volume form for the Euclidean structure on E. Let dA_s be the two-form on $M \setminus \mathcal{Z}$ given by the pushforward of ω along f.

For every $\varepsilon > 0$ let $M_\varepsilon = \{q \in M \mid d(q, \mathcal{Z}) > \varepsilon\}$, where $d(\cdot, \cdot)$ is the almost-Riemannian distance (see equation (1.4)). We say that K is $\mathcal{S}$-*integrable* if

$$\lim_{\varepsilon \to 0} \int_{M_\varepsilon} K \, dA_s$$

exists and is finite. In this case we denote such limit by $\int_M K dA_s$.

When $\mathcal{S}$ has no tangency points K happens to be $\mathcal{S}$-integrable and $\int_M K dA_s$ is determined by the topology of M^+ and M^-. This result, appeared in [3] and recalled in Theorem 2.5, can be seen as a generalization of Gauss–Bonnet formula to ARSs.

Theorem 2.5 ([3]). *Let M be a compact oriented surface endowed with an oriented 2-ARS $\mathcal{S}$ for which condition (H0) holds true. Assume that $\mathcal{S}$ has no tangency points. Then K is $\mathcal{S}$-integrable and*

$$\int_M K dA_s = 2\pi(\chi(M^+) - \chi(M^-)). \qquad (2.8)$$

Remark that, by Theorem 2.1, equation (2.8) becomes $\int_M K dA_s = 2\pi e(E)$.

The proof of Theorem 2.5 relies on the fact that if there are not tangency points then the boundary of M_ε is smooth for every ε in a (right) neighborhood of zero. This allow the authors in [3] to apply the Gauss–Bonnet formula for Riemannian structures on M_ε. Then, computing the integral of the geodesic curvature on the boundary of M_ε in a neighborhood of a Grushin point, they show that the two terms from different sides of $\mathcal{Z}$ offset each other as ε goes to zero. Finally they conclude using the compactness of $\mathcal{Z}$.

2.2.1 $\mathcal{S}$-integrability in presence of tangency points: numerical simulations

As concerns the notion of integrability of the curvature with respect to the Riemannian density on $M \setminus \mathcal{Z}$, it turns out that the hypothesis made in Theorem 2.5 about the absence of tangency points is not just technical. Indeed, in this section we provide some numerical simulations hinting that, when $\mathcal{T} \neq \emptyset$,

$$\int_{M_\varepsilon} K dA_s$$

does not converge, in general, as ε tends to zero.

From the proof of Theorem 2.5 we know that far from tangency points the integral of the geodesic curvature along ∂M_ε^+ and ∂M_ε^- offset each other for ε going to zero, where $M_\varepsilon^\pm = M^\pm \cap M_\varepsilon$. Hence, to understand whether the presence of a tangency point may lead to non-$\mathcal{S}$-integrability of K it is sufficient to compute the geodesic curvature of ∂M_ε^+ and ∂M_ε^- in a neighborhood of such a point. More precisely consider the almost-Riemannian structure $(E, f, \langle \cdot, \cdot \rangle)$ on $M = \mathbb{R}^2$ for which $E = \mathbb{R}^2 \times \mathbb{R}^2$, $f((x, y), (a, b)) = ((x, y), (a, b(y - x^2)))$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product. For this system one has

$$K = \frac{-2 \left(3 x^2 + y \right)}{\left(x^2 - y \right)^2}.$$

The graph of K is illustrated in Figure 2.2. Notice that $\limsup_{q \to (0,0)} K(q) = +\infty$ and $\liminf_{q \to (0,0)} K(q) = -\infty$. This situation is different from the Grushin case where $K(q)$ diverges to $-\infty$ as q approaches $\mathcal{Z}$.

For every $\varepsilon > 0$, the sets ∂M_ε^+ and ∂M_ε^- are smooth manifolds except at their intersections with the vertical axis $x = 0$, which is the cut locus for the problem of minimizing the distance from $\mathcal{Z} = \{(x, x^2) \mid x \in \mathbb{R}\}$. Fix $0 < a < 1$ and consider the two geodesics starting from the point (a, a^2) and minimizing (locally) the distance from $\mathcal{Z}$. Let P^+ and P^- be the two points along these geodesics at
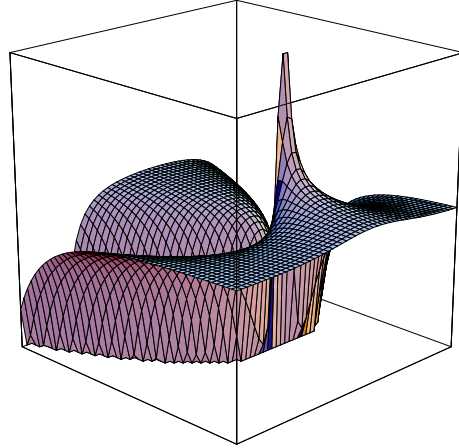
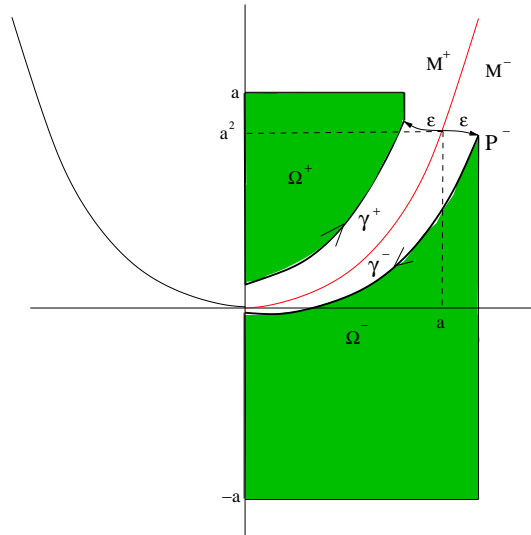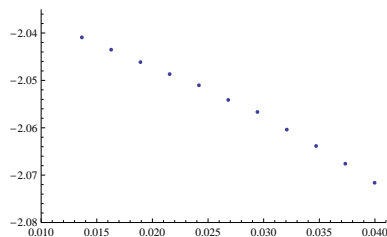Figure 2.2: Graph of K for $\Delta = \text{span}((1,0),(0,y-x^2))$

Figure 2.3: Regions $\Omega^{\pm}$ where to apply Riemannian Gauss–Bonnet formula

Figure 2.4: Divergence of the $\mathcal{S}$-integral of K

distance ε from $\mathcal{Z}$. Denote by γ^+ and γ^- the portions of ∂M_ε^+ and ∂M_ε^- connecting the vertical axis to the points P^+ and P^-, oriented as in Figure 2.3. It is easy to approximate numerically γ^+ and γ^- by broken lines, but the evaluation of the integral of their geodesic curvatures is very unstable since its computation involves the second derivative of the curve parameterized by arclength. To avoid this problem, we rather apply the Riemannian Gauss–Bonnet formula on the regions Ω^+ and Ω^- introduced in Figure 2.3. This works better since the integral of the Gaussian curvature on Ω^+ and Ω^- is numerically stable, and the integral of the geodesic curvature on horizontal and vertical segments can be computed analytically (in particular it is always zero on horizontal segments). Figure 2.4 shows the value of

$$\varepsilon \left(\int_{\gamma^+} K_g ds - \int_{\gamma^-} K_g ds \right)$$

for $a = 0.1$ and ε varying in the interval $[0.01, 0.04]$. The graph seems to converge as ε tends to zero to a nonzero constant, strongly hinting at the divergence of $\int_{M_\varepsilon} K dA_s$.

2.2.2 More general notion of $\mathcal{S}$-integrability and statement of the result

One possible explanation of the fact that the integral $\int_{M_\varepsilon} K dA_s$ seems to diverge when $\mathcal{T} \neq \emptyset$ is the interaction between different orders in the asymptotic expansion of the almost-Riemannian distance. To avoid this interference, we define a 3-scale integral of the curvature.

Definition 2.1. (3-scale $\mathcal{S}$-integrability) Let $q \in \mathcal{T}$ and U^q be a neighborhood of q such that an orthonormal frame for G on U^q is given by the local representation (F3). For δ_1, $\delta_2 > 0$ sufficiently small the rectangle $[-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$ is a subset of U^q denoted by $B^q_{\delta_1, \delta_2}$. For every $\varepsilon > 0$, define

$$M_{\varepsilon, \delta_1, \delta_2} = M_\varepsilon \setminus \bigcup_{q \in \mathcal{T}} B^q_{\delta_1, \delta_2}.$$

We say that K is 3-scale $\mathcal{S}$-integrable if

$$\lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} \lim_{\varepsilon \to 0} \int_{M_{\varepsilon, \delta_1, \delta_2}} K dA_s \qquad (2.9)$$

exists, is finite and does not depend on the choice of the local representation. In this case we denote such limit by $\oiint_M K dA_s$.
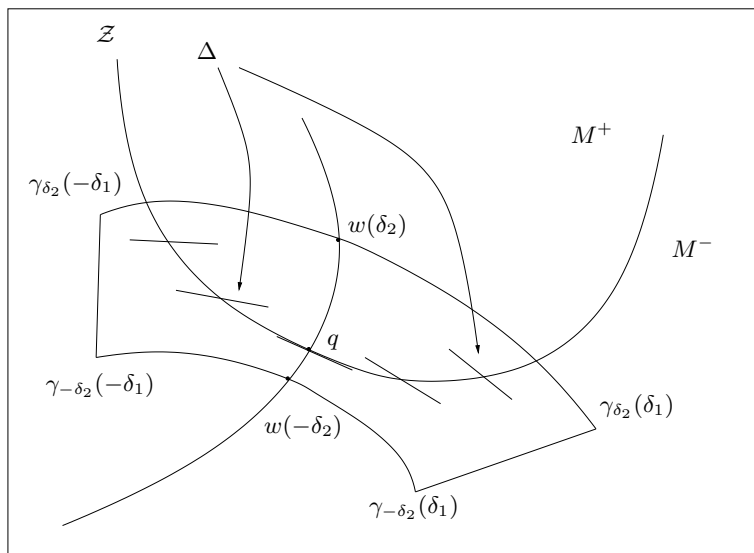
Figure 2.5: The rectangular box $B^q_{\delta_1,\delta_2}$

Remark 2.3. Notice that if $\mathcal{T} = \emptyset$, then the concepts of $\mathcal{S}$-integrability and 3-scale $\mathcal{S}$-integrability coincide. Moreover, the order in which the limits are taken in (2.9) is important. Indeed, if the order is permuted, the result given in Theorem 2.6 does not hold anymore.

Remark 2.4. Thanks to Theorem 3.4 (see next chapter), at each tangency point q we have a canonical choice for the coordinate system used to define the rectangular box $B^q_{\delta_1,\delta_2}$ whence Definition 2.1 is intrinsic.

The following result is a generalization of the classical Gauss–Bonnet formula for Riemannian structures to generic oriented two-dimensional almost-Riemannian structures.

Theorem 2.6. *Let M be a compact oriented surface. If an oriented almost-Riemannian structure $\mathcal{S}$ on M satisfies the hypothesis* **(H0)***, then K is 3-scale $\mathcal{S}$-integrable and*

$$\oiint_M K dA_s = 2\pi e(E). \tag{2.10}$$

Notice that the right-hand side of formula (2.10) does not depend on the choice of coordinate systems around tangency points made in Definition 2.1.

As a consequence of Theorems 2.1 and 2.6 we get the following corollary.

Corollary 2.7. *Let M be a compact oriented surface. For an oriented almost-Riemannian structure $\mathcal{S}$ on M satisfying the generic hypothesis (H0) we have that*

$\mathcal{S}$ *is trivializable if and only if* $\oiint_M K dA_s = 0$. *In particular, if* $\mathcal{S}$ *has not tangency points then* $\int_M K dA_s = 0$ *if and only if* $\mathcal{S}$ *is trivializable.*

These results show the relation between the integral of the curvature and the topology of the manifold for two-dimensional almost-Riemannian structures.

2.2.3 Proof of Theorem 2.6

Let us recall the following Gauss–Bonnet-like formula for domains whose boundary is $\mathcal{C}^2$ in a neighborhood of $\mathcal{Z}$.

Theorem 2.8 ([17], Theorem 5.2)**.** *Let U be an open bounded connected subset of M such that* **i)** $\overline{U}$ *contains only ordinary and Grushin points,* **ii)** ∂U *is piecewise $\mathcal{C}^2$,* **iii)** ∂U *is the union of the supports of a finite set of admissible curves $\gamma^1, \ldots, \gamma^m$,* **iv)** ∂U *is $\mathcal{C}^2$ in a neighborhood of $\mathcal{Z}$.*

Define $U_\varepsilon^\pm = M_\varepsilon^\pm \cap U$. Then the following limits exist and are finite

$$\int_U K dA_s := \lim_{\varepsilon \to 0} \int_{U_\varepsilon^+ \cup U_\varepsilon^-} K dA_s,$$

$$\int_{\partial U} k_g d\sigma_s := \lim_{\varepsilon \to 0} \left(\int_{\partial U \cap \partial U_\varepsilon^+} k_g d\sigma - \int_{\partial U \cap \partial U_\varepsilon^-} k_g d\sigma \right),$$

where we interpret each integral $\int_{\partial U \cap \partial U_\varepsilon^\pm} k_g d\sigma$ as the sum of the integrals along the $\mathcal{C}^2$ portions of $\partial U \cap \partial U_\varepsilon^\pm$, plus the sum of the angles at the points of $\partial U \cap \partial U_\varepsilon^\pm$ where ∂U is not $\mathcal{C}^1$. Moreover, we have

$$\int_U K dA_s + \int_{\partial U} k_g d\sigma_s = 2\pi(\chi(U^+) - \chi(U^-)).$$

Fix δ_1 and δ_2 in such a way that the rectangles B_{δ_1,δ_2}^q are pairwise disjoint and $\mathcal{Z} \cap \partial B_{\delta_1,\delta_2}^q \subset [-\delta_1, \delta_1] \times \{\delta_2\}$, for every $q \in \mathcal{T}$. By construction, $\partial B_{\delta_1,\delta_2}^q$ is the union of four admissible curves for every $q \in \mathcal{T}$. Applying Theorem 2.8 with $U = M_\varepsilon \setminus \bigcup_{q \in \mathcal{T}} B_{\delta_1,\delta_2}^q$, we have

$$\lim_{\varepsilon \to 0} \int_{M_{\varepsilon,\delta_1,\delta_2}} K dA_s + \sum_{q \in \mathcal{T}} \int_{\partial B_{\delta_1,\delta_2}^q} k_g d\sigma_s = 2\pi(\chi(M^+ \setminus \bigcup_{q \in \mathcal{T}} B_{\delta_1,\delta_2}^q) - \chi(M^- \setminus \bigcup_{q \in \mathcal{T}} B_{\delta_1,\delta_2}^q))$$

$$= 2\pi(\chi(M^+) - \chi(M^-)) = 2\pi(e(E) - \tau(\mathcal{S})),$$

where the last equality follows from Theorem 2.1. We are left to prove that, for a fixed $q \in \mathcal{T}$,

$$\lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} \int_{\partial B_{\delta_1,\delta_2}^q} k_g d\sigma_s = -2\pi\tau_q.$$

To show the last equality, take a local representation of the type $(F3)$ and assume that $M^+ \cap U^q$ is the set $\{y - x^2\psi(x) < 0\} \cap U^q$, the proof for the opposite situation being analogous. On one hand, one can check that $\tau_q = 1$. On the other hand, the

geodesic curvature along $[-\delta_1, \delta_1] \times \{\delta_2\}$ and along $[-\delta_1, \delta_1] \times \{-\delta_2\}$ is zero, the two segments being the support of geodesics. Hence

$$\int_{\partial B^q_{\delta_1, \delta_2}} k_g d\sigma_s = \int_{\{\delta_1\} \times [-\delta_2, \delta_2]} k_g d\sigma_s + \int_{\{-\delta_1\} \times [-\delta_2, \delta_2]} k_g d\sigma_s + \sum_{j=1}^{4} \alpha_j$$

where the last term is the sum of the values of the angles of the box and is equal to -2π. Indeed, because of the diagonal form of the metric with respect to the chosen coordinates, each angle has value $-\frac{\pi}{2}$. The first two terms are well defined and tend to zero when δ_2 tends to zero. Hence

$$\lim_{\delta_1 \to 0} \lim_{\delta_2 \to 0} \int_{\partial B^q_{\delta_1, \delta_2}} k_g d\sigma_s = -2\pi = -2\pi\tau_q.$$

∎

2.3 Lipschitz Equivalence

Let M_1, M_2 be two manifolds. For $i = 1, 2$, let $\mathcal{S}_i = (E_i, f_i, \langle \cdot, \cdot \rangle_i)$ be a rank-varying sub-Riemannian structure on M_i. Denote by d_i the Carnot–Caratheodory distance on M_i associated with $\mathcal{S}_i$.

Definition 2.2. We say that a diffeomorphism $\varphi : M_1 \to M_2$ is a *Lipschitz equivalence* if it is bi-Lipschitz as a map from (M_1, d_1) to (M_2, d_2).

Let us recall that considering the subclass of Riemannian structures, Lipschitz equivalence coincides with differential equivalence. That is, two Riemannian structures on two compact manifolds are Lipschitz equivalent if and only if the two manifolds are diffeomorphic. In the following we consider this problem for (generic) oriented almost-Riemannian structures on compact oriented surfaces. It is well-known that all the differential structures on a given surface are equivalent, hence the question we are going to face is wether there exist two (genuinely) almost-Riemannian structures on two diffeomorphic compact oriented surfaces that are not Lipschitz equivalent. As we shall see, the answer is positive: the Lipschitz equivalence class of a 2-ARS changes depending on how the singular set splits the surface and on how the tangency points are located on the singular set.

2.3.1 Graph of a 2-ARS

Let M be a compact oriented surface and $\mathcal{S}$ be an oriented ARS on M satisfying (H0).

We associate with $\mathcal{S}$ the graph $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ where

- each vertex in $\mathcal{V}(\mathcal{G})$ represents a connected component of $M \setminus \mathcal{Z}$;

- each edge in $\mathcal{E}(\mathcal{G})$ represents a connected component of $\mathcal{Z}$;

- the edge corresponding to a connected component W connects the two vertices corresponding to the connected components M_1 and M_2 of $M \setminus \mathcal{Z}$ such that $W \subset \partial M_1 \cap \partial M_2$.

Thanks to the hypothesis (H0), every connected component of $\mathcal{Z}$ joints a connected component of M^+ and one of M^-. Thus the graph $\mathcal{G}$ turns out to be *bipartite*, i.e., there exists a partition of the set of vertices into two subsets V^+ and V^- such that each edge of $\mathcal{G}$ joins a vertex of V^+ to a vertex of V^-. Conversely, it is not difficult to see that every finite bipartite graph can be obtained from an oriented 2-ARS (satisfying (H0)) on a compact oriented surface.

Using the bipartite nature of $\mathcal{G}$ we introduce an orientation on $\mathcal{G}$ given by two functions $\alpha, \omega : \mathcal{E}(\mathcal{G}) \to \mathcal{V}(\mathcal{G})$ defined as follows. If e corresponds to W then $\alpha(e) = v$ and $\omega(e) = w$, where v and w correspond respectively to the connected components $M_v \subset M^-$ and $M_w \subset M^+$ such that $W \subseteq \partial M_v \cap \partial M_w$.

We label each vertex v corresponding to a connected component $\hat{M}$ of $M \setminus \mathcal{Z}$ with a pair $(\text{sign}(v), \chi(v))$ where $\text{sign}(v) = \pm 1$ if $\hat{M} \subset M^{\pm}$ and $\chi(v)$ is the Euler characteristic of $\hat{M}$. We define for every $e \in E(\mathcal{G})$ the number $\tau(e) = \sum_{q \in W \cap \mathcal{T}} \tau_q$, where W is the connected component of $\mathcal{Z}$ corresponding to e.

Finally, we define a label for each edge e corresponding to a connected component W of $\mathcal{Z}$ containing tangency points. Let $s \geq 1$ be the cardinality of the set $W \cap \mathcal{T}$. The label of e is an equivalence class of s-uples with entries in $\{\pm 1\}$ defined as follows. Fix on W the orientation induced by M^+ and choose a point $q \in W \cap \mathcal{T}$. Let $q_1 = q$ and for every $i = 1, \ldots, s - 1$ let q_{i+1} be the first element in $W \cap \mathcal{T}$ that we meet after q_i walking along W in the fixed orientation. We associate with
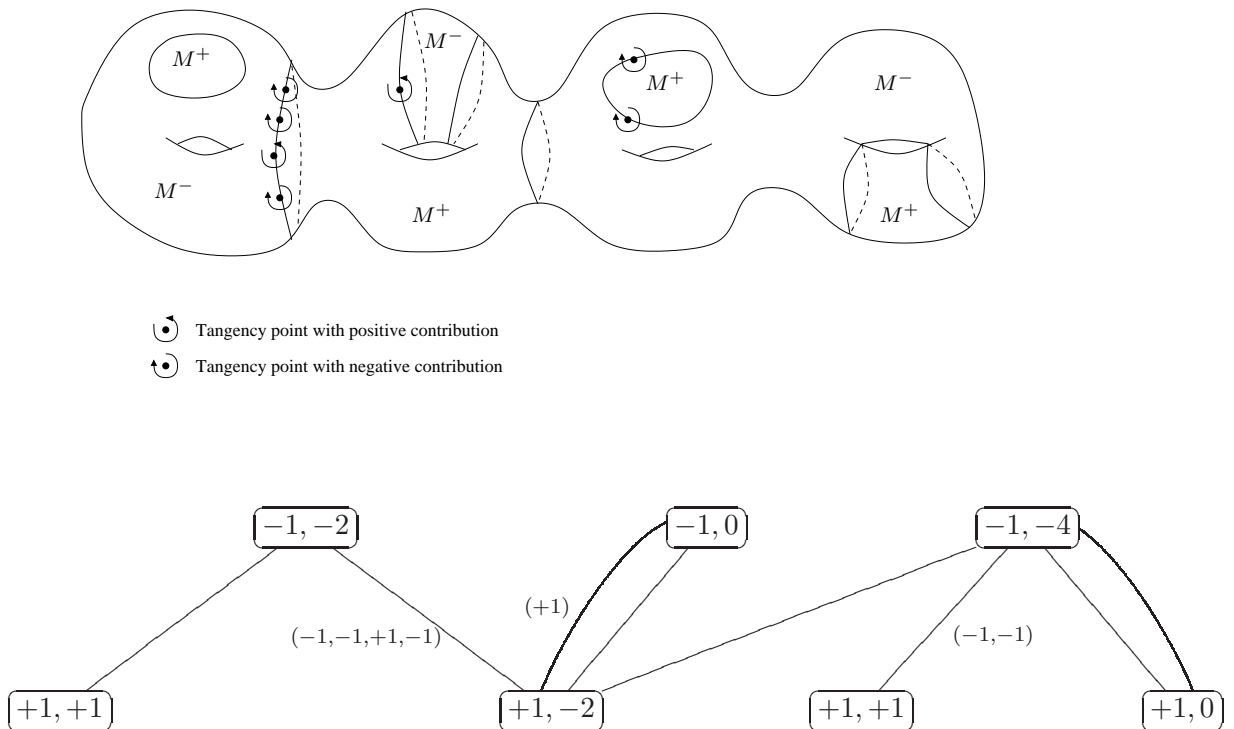

Figure 2.6: Example of ARS on a surface of genus 4 and corresponding labelled graph

e the equivalence class of $(\tau_{q_1}, \tau_{q_2}, \ldots, \tau_{q_s})$ in the set of s-uples with entries in $\{\pm 1\}$

modulo cyclic permutations. In Figure 2.7 an ARS on a surface of genus 4 and its labelled graph (Figure 2.7(a)) are portrayed. According to our definition of labels on edges, Figures 2.7(a) and 2.7(b) represent equal graphs associated with the same ARS. On the other hand, the graph in Figure 2.7(c) is not the graph associated to the ARS of Figure 2.7. In Figure 2.8 two steps in the construction of the labelled graph associated with the ARS in Figure 2.6 are shown.

Remark 2.5. Once an orientation on E is fixed the labelled graph associated with $\mathcal{S}$ is unique.

Tangency point with positive contribution

Tagency point with negative contribution

(a) (b)

(c)
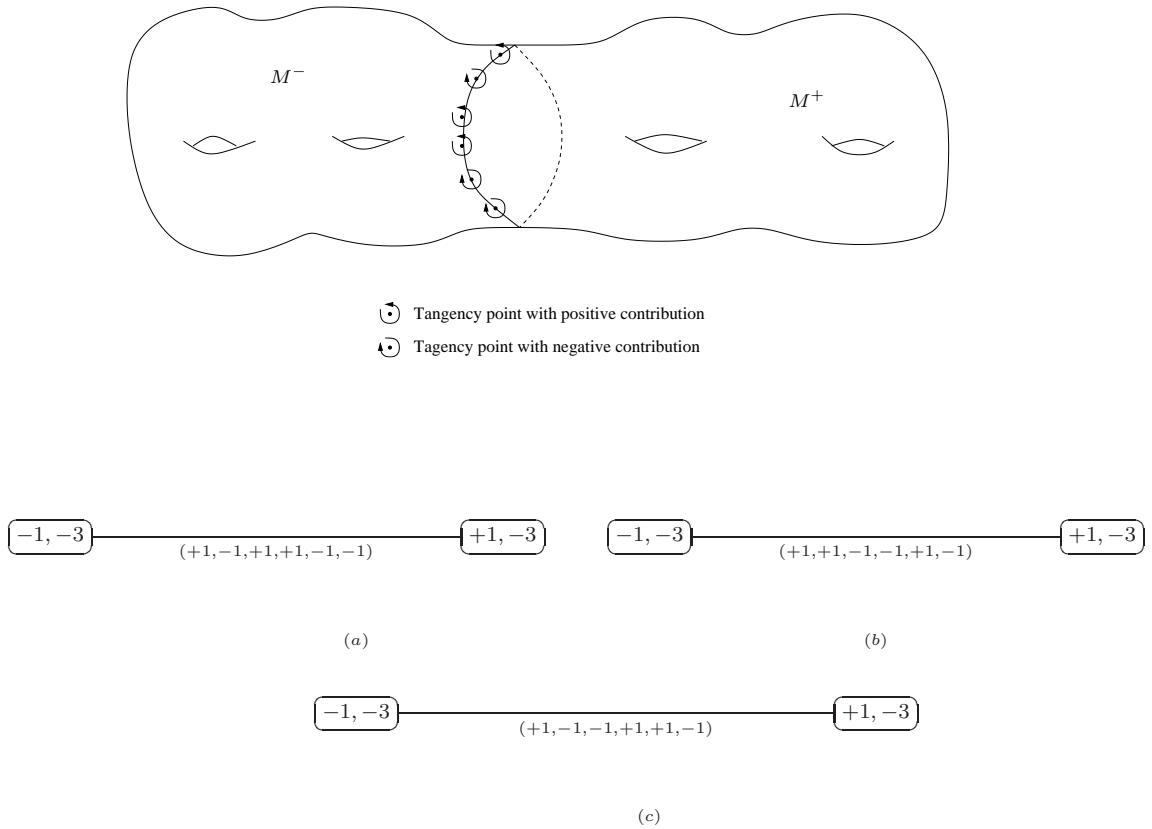
Figure 2.7: Example of ARS on a surface of genus 4. Figures (a) and (b) illustrate equal labelled graphs associated with the ARS. Figure (c) gives an example of labelled graph different from the graph in Figure (a)

We define an equivalence relation on the set of graphs associated with oriented ARS on M satisfying hypothesis (H0).

Definition 2.3. Let $\mathcal{S}_i = (E_i, f_i, \langle \cdot, \cdot \rangle_i)$ be an oriented almost-Riemannian structure on a compact oriented surface M_i, $i = 1, 2$. Assume that $\mathcal{S}_i$ satisfies hypothesis (H0), $i = 1, 2$. Let $\mathcal{G}_i$ be the labelled graph associated with $\mathcal{S}_i$ and denote by $\alpha_i, \omega_i : \mathcal{E}(\mathcal{G}_i) \to \mathcal{V}(\mathcal{G}_i)$ the functions defined as above. We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ have *equivalent graphs* if , after possibly changing the orientation on E_2, they have the same labelled graph.
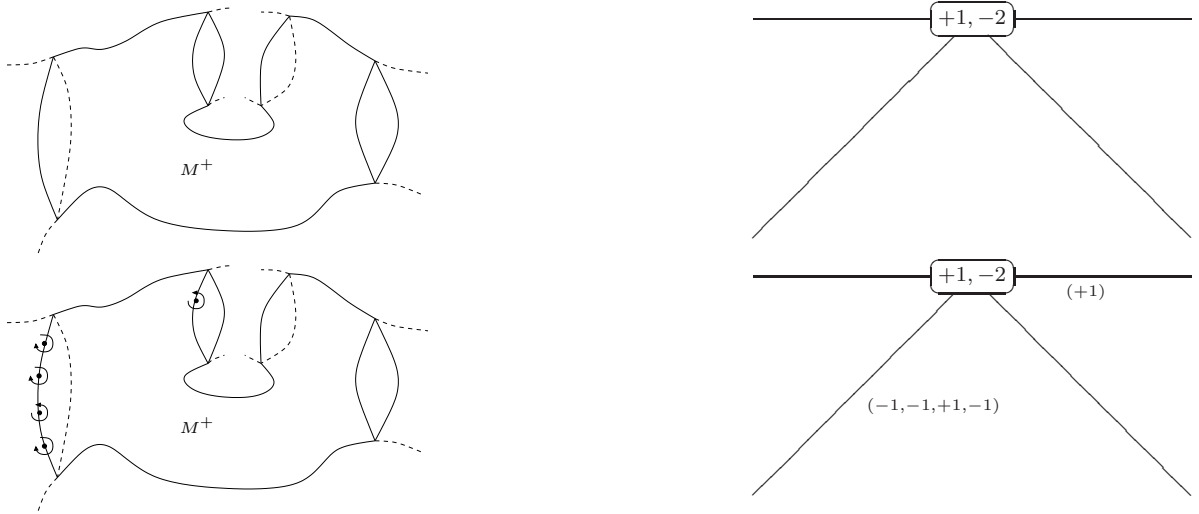
Figure 2.8: Algorythm to build the graph

In other words, after possibly changing the orientation on E_2 and still denoting by $\mathcal{G}_2$ the associated graph, there exist bijections $u : \mathcal{V}(\mathcal{G}_1) \to \mathcal{V}(\mathcal{G}_2)$, $k : \mathcal{E}(\mathcal{G}_1) \to \mathcal{E}(\mathcal{G}_2)$ such that the diagram

$$\begin{array}{ccc}
\mathcal{V}(\mathcal{G}_1) & \xrightarrow{\ u\ } & \mathcal{V}(\mathcal{G}_2) \\
{\scriptstyle \alpha_1}\big\uparrow & & \big\uparrow{\scriptstyle \alpha_2} \\
\mathcal{E}(\mathcal{G}_1) & \xrightarrow{\ k\ } & \mathcal{E}(\mathcal{G}_2)
\end{array} \qquad (2.11)$$

commutes and u and k preserve labels.

Figure 2.9 illustrates the graph associated with the ARS obtained by reversing the orientation of the ARS in Figure 2.6.
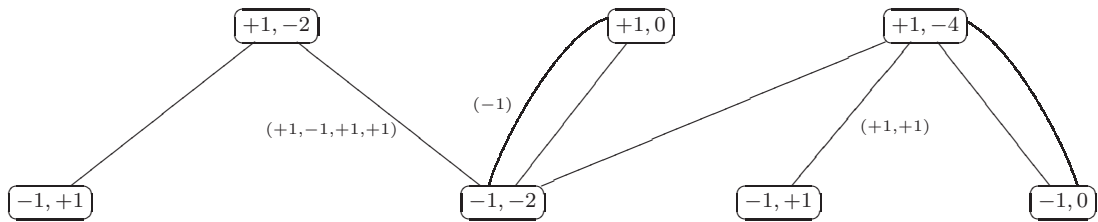
Figure 2.9: Equivalent graph to the one in Figure 2.6

2.3.2   Statement of the main result and useful facts

Let us give a classification of oriented 2-ARS using the labelled graph defined in previous section.

Theorem 2.9. *Two oriented almost-Riemannian structures, defined on compact oriented surfaces and satisfying (H0), are Lipschitz equivalent if and only if they have equivalent graphs.*

This theorem shows another interesting difference between Riemannian manifolds and almost-Riemannian ones: in the Riemannian context, Lipschitz equivalence coincides with the equivalence as differentiable manifolds; in the almost-Riemannian context, Lipschitz equivalence is a stronger condition. Notice, however, that in general Liptschitz equivalence does not imply isometry. Indeed, the Lipschitz equivalence between two structures does not depend on the metric structure but only on the submodule Δ. This is highlightened by the fact that the graph itself depends only on Δ.

Before proving Theorem 2.9, we show two general propositions that will be useful in the following. The first one essentially says that the underlying bundles of two rank-varying sub-Riemannian structures having the same submodule are isomorphic.

Proposition 2.10. *Given two (n, k)-rank-varying distributions $(E_i, f_i), i = 1, 2$ on the same manifold M, assume that they define the same submodule of $\mathrm{Vec}(M)$, i.e.,*

$$(f_1)_*(\Gamma(E_1)) = (f_2)_*(\Gamma(E_2)) = \Delta \subseteq \mathrm{Vec}(M).$$

Then, there exists an isomorphism of vector bundles $h : E_1 \to E_2$ such that $f_2 \circ h = f_1$.

Proof. Since $(f_i)_* : \Gamma(E_i) \to \Delta, i = 1, 2$, are isomorphisms of $\mathcal{C}^\infty(M)$-modules, then $(f_2)_*^{-1} \circ (f_1)_* : \Gamma(E_1) \to \Gamma(E_2)$ is an isomorphism. A classical result given in [26, Proposition XIII p.78] states that the map $f \mapsto f_*$ is an isomorphism of $\mathcal{C}^\infty(M)$-modules from the set of morphisms from E_1 to E_2 to the set of morphisms from $\Gamma(E_1)$ to $\Gamma(E_2)$. Applying this result, there exists a unique isomorphism $h : E_1 \to E_2$ such that $h_* = (f_2)_*^{-1} \circ (f_1)_*$. By construction, $(f_2)_* \circ h_* = (f_1)_*$ and applying again [26, Proposition XIII p.78] we get $f_2 \circ h = f_1$. ∎

We give now a characterization of admissible curves (see Definition 1.4) as curves that are Lipschitz continuous with respect to the Carnot–Caratheodory distance. Notice that the following proposition holds in the context of rank-varying sub-Riemannian structures.

Proposition 2.11. *Let $(E, f, \langle \cdot, \cdot \rangle)$ be a rank-varying sub-Riemannian structure on a manifold M. Let $\gamma : [0, T] \to M$ be an absolutely continuous curve. Then γ is admissible if and only if it is Lipschitz continuous with respect to the sub-Riemannian distance.*

Proof. First we prove that if the curve is admissible then it is Lipschitz with respect to d (*d-Lipschitz* for short). This is a direct consequence of the definition of the sub-Riemannian distance. Indeed, let

$$[0, T] \ni t \mapsto u(t) \in E_{\gamma(t)}$$

be a control function for γ and let $L > 0$ be the essential supremum of $\sqrt{\langle u, u \rangle}$ on $[0, T]$. Then, for every subinterval $[t_0, t_1] \subset [0, T]$ one has

$$d(\gamma(t_0), \gamma(t_1)) \leq \int_{t_0}^{t_1} \sqrt{\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t))} dt \leq \int_{t_0}^{t_1} \sqrt{\langle u(t), u(t) \rangle} dt \leq L(t_1 - t_0).$$

Hence γ is d-Lipschitz.

Viceversa, assume that γ is d-Lipschitz with Lipschitz constant L. Since γ is absolutely continuous, it is differentiable almost everywhere on $[0, T]$. Thanks to the Ball-Box Theorem (see [9]), for every $t \in [0, T]$ such that the tangent vector $\dot{\gamma}(t)$ exists, $\dot{\gamma}(t)$ belongs to the distribution $\Delta(\gamma(t))$ (if not, the curve would fail to be d-Lipschitz). Hence for almost every $t \in [0, T]$ there exists $u_t \in E_{\gamma(t)}$ such that $\dot{\gamma}(t) = f(u_t)$. Moreover, since the curve is d-Lipschitz, one has that $\mathbf{G}_{\gamma(t)}(\dot{\gamma}(t)) \leq L^2$ for almost every $t \in [0, T]$. This can be seen computing lengths in privileged coordinates (see [9] for the definition of this system of coordinates). Hence, we can assume that $\langle u_t, u_t \rangle \leq L^2$ almost everywhere. Finally, we apply Filippov Theorem (see [18, Theorem 3.1.1 p.36]) to the differential inclusion

$$\dot{\gamma}(t) \in \{f(u) \mid \pi_E(u) = \gamma(t) \text{ and } \langle u, u \rangle \leq L^2\}.$$

that assures the existence of a measurable choice of the control function corresponding to γ. Thus γ is admissible. ∎

2.3.3 Proof of Theorem 2.9

First of all, we notice that we can assume $M_1 = M_2 = M$. Indeed, if two ARSs are Lipschitz equivalent, then by definition there exists a diffeomorphism $\varphi : M_1 \to M_2$. On the other hand, if the associated graphs are equivalent, then, summing the second entries of the labels of all vertices, we find $\chi(M_1) = \chi(M_2)$, i.e., M_1 and M_2 are diffeomorphic.

Necessity

We show that if $\mathcal{S}_1, \mathcal{S}_2$ are Lipschitz equivalent then their graphs are equivalent.

Denote by M_i^+, respectively M_i^-, the set where f_i is an orientation-preserving, respectively orientation-reversing, isomorphism of vector bundles, and by Δ^i the submodule $\{f_i \circ \sigma \mid \sigma \in \Gamma(E_i)\}$. Let $\mathcal{Z}_i$ be the singular locus of $\mathcal{S}_i$ and $\mathcal{T}_i$ the set of tangency points of $\mathcal{S}_i$. Finally, for every $q \in \mathcal{T}_i$, denote by τ_q^i the contribution at the tangency point defined in Section 2.1.1 with $\Delta = \Delta^i$.

In this section we assume $\varphi : (M, d_1) \to (M, d_2)$ to be a Lipschitz equivalence and we show that $\mathcal{S}_1$ and $\mathcal{S}_2$ have equivalent graphs. As a consequence of the Ball-Box Theorem (see, for instance, [9]) one can prove the following result.

Lemma 2.12. *If p is an ordinary, Grushin or tangency point for $\mathcal{S}_1$, then $\varphi(p)$ is an ordinary, Grushin or tangency point for $\mathcal{S}_2$, respectively.*

Thanks to Lemma 2.12, for every connected component $\hat{M}$ of $M \setminus \mathcal{Z}_1$, $\varphi(\hat{M})$ is a connected component of $M \setminus \mathcal{Z}_2$ and for every connected component W of $\mathcal{Z}_1 \cap \partial \hat{M}$, $\varphi(W)$ is a connected component of $\mathcal{Z}_2 \cap \partial \varphi(\hat{M})$. Moreover, since $\varphi|_{\overline{\hat{M}}}$ is a diffeomorphism, it follows that $\chi(\hat{M}) = \chi(\varphi(\hat{M}))$. After possibly changing the orientation on E_2, we may assume $\varphi(M_1^\pm) = M_2^\pm$. We will prove that, in this case, the labelled graphs are equal. Indeed, if $v \in \mathcal{V}(\mathcal{G}_1)$ corresponds to $\hat{M}$, define $u(v) \in \mathcal{V}(\mathcal{G}_2)$ as the vertex corresponding to $\varphi(\hat{M})$. If $e \in \mathcal{E}(\mathcal{G}_1)$ corresponds to

W define $k(e) \in \mathcal{E}(\mathcal{G}_2)$ as the edge corresponding to $\varphi(W)$. Then $\chi(u(v)) = \chi(v)$, $\mathrm{sign}(u(v)) = \mathrm{sign}(v)$, and, by construction, the diagram (2.11) commutes.

Let us compute the contribution at a tangency point q of an ARS $(E, f, \langle \cdot, \cdot \rangle)$ using the corresponding local representation given in Theorem 1.3.

Lemma 2.13. *Let $\gamma : [0, T] \to M$ be a smooth curve such that $\gamma(0) = q \in \mathcal{T}$ and $\dot{\gamma}(0) \in \Delta(q) \setminus \{0\}$. Assume moreover that γ is d-Lipschitz, where d is the almost-Riemannian distance, and that $\gamma((0, T))$ is contained in one of the two connected components of $M \setminus \mathcal{Z}$. Let (x, y) be a coordinate system centered at q such that the local representation (F3) of Theorem 1.3 applies. Then $\gamma((0, T)) \subset \{(x, y) \mid y - x^2\psi(x) < 0\}$. Moreover, if $\{(x, y) \mid y - x^2\psi(x) < 0\} \subseteq M^+$, resp. M^-, then $\tau_q = 1$, resp. -1.*

Proof. Since $\gamma(0) = (0, 0)$ and $\dot{\gamma}(0) \in \mathrm{span}\{(1, 0)\} \setminus \{0\}$, there exist two smooth functions $\overline{x}(t), \overline{y}(t)$ such that $\gamma(t) = (t\overline{x}(t), t^2\overline{y}(t))$ and $\overline{x}(0) \neq 0$. Assume by contradiction that $\gamma((0, T)) \subset \{(x, y) \mid y - x^2\psi(x) > 0\}$, i.e., for $t \in (0, T)$, $\overline{y}(t) > \psi(t\overline{x}(t))\overline{x}(t)^2$. Since $\psi(0) > 0$, for t sufficiently small $\psi(t\overline{x}(t)) > 0$ and $\overline{y}(t)^{1/3} > \psi(t\overline{x}(t))^{1/3}|\overline{x}(t)|^{2/3}$. By the Ball-Box Theorem (see [9]) there exist c_1, c_2 positive constants such that, for t sufficiently small we have

$$c_1(|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3}) \leq d(\gamma(t), (0, 0)) \leq c_2(|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3}).$$

On the other hand, for t sufficiently small,

$$|t\overline{x}(t)| + |t^2\overline{y}(t)|^{1/3} > t^{2/3}|\overline{x}(t)|^{2/3}\psi(t\overline{x}(t))^{1/3}.$$

Hence, for t sufficiently small, $d(\gamma(t), (0, 0)) > c_3 t^{2/3}$, with $c_3 > 0$. This implies that γ is not Lipschitz with respect to the almost-Riemannian distance. Finally, a direct computation shows the assertion concerning τ_q, see also Figure 2.1. $\blacksquare$

Next lemma, jointly with Lemma 2.12, guarantees that the two bijections u and k preserve labels.

Lemma 2.14. *Let $q \in \mathcal{T}_1$. Then $\tau_q^1 = \tau_{\varphi(q)}^2$.*

Proof. Apply Theorem 1.3 to $\mathcal{S}_1$ and find a neighborhood U of q and a coordinate system (x, y) on U such that $q = (0, 0)$ and $\mathcal{Z}_1 \cap U = \{(x, y) \mid y = x^2\psi(x)\}$. Let $\sigma, \rho \in \Gamma(E|_U)$ be the local orthonormal frame such that $f_1 \circ \sigma = X$ and $f_1 \circ \rho = Y$. Assume that $U_1^+ = M_1^+ \cap U = \{(x, y) \mid y - x^2\psi(x) > 0\}$. Fix $T > 0$ and consider the smooth curve $\gamma : [0, T] \to U$ defined by $\gamma(t) = (t, 0)$. Then γ is admissible for $\mathcal{S}_1$ with control function $u(t) = \sigma(t, 0)$. By definition, for T sufficiently small $\gamma((0, T))$ lies in a single connected component of $U \setminus \mathcal{Z}_1$. Moreover, by Proposition 2.11, γ is a d_1-Lipschitz map with Lipschitz constant less or equal to 1. Hence, according to Lemma 2.13, $\tau_q^1 = -1$.

Consider the curve $\tilde{\gamma} = \varphi \circ \gamma : [0, T] \to \varphi(U)$. Since φ is Lipschitz, $\tilde{\gamma}$ is d_2-Lipschitz as a map from the interval $[0, T]$ to the metric space $(\varphi(U), d_2)$. Moreover, $\tilde{\gamma}$ is smooth and $\dot{\tilde{\gamma}}(0) \in \Delta^2(\varphi(q)) \setminus \{0\}$, φ being a diffeomorphism mapping $\mathcal{Z}_1$ to $\mathcal{Z}_2$. Finally, since $\varphi(M_1^-) = M_2^-$, then $\tilde{\gamma}((0, T)) \subset U_2^- = \varphi(U) \cap M_2^-$. Thus, by Lemma 2.13, $\tau_{\varphi(q)}^2 = -1$. Analogously, one can prove the statement in the case $U_1^+ = \{(x, y) \mid y - x^2\psi(x) < 0\}$ (for which $\tau_q^1 = \tau_q^2 = 1$). $\blacksquare$

Lemma 2.14 implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ have equal labelled graphs. This concludes the proof that having equivalent graphs is a necessary condition for two ARSs being Lipschitz equivalent.

Sufficiency

In this section we prove that if $\mathcal{S}_1$ and $\mathcal{S}_2$ have equivalent graphs then there exists a Lipschitz equivalence between (M, d_1) and (M, d_2). After possibly changing the orientation on E_2, we assume the associated labelled graphs to be equal, i.e., there exist two bijections u, k as in Definition 2.3 such that diagram (2.11) commutes.

The proof is in five steps. The first step consists in proving that we may assume $E_1 = E_2$. The second step shows that we can restrict to the case $\mathcal{Z}_1 = \mathcal{Z}_2$ and $\mathcal{T}_1 = \mathcal{T}_2$. In the third step we prove that we can assume that $\Delta^1(q) = \Delta^2(q)$ at each point $q \in M$. As fourth step, we demonstrate that the submodules Δ^1 and Δ^2 coincide. In the fifth and final step we remark that we can assume $f_1 = f_2$ and conclude. The Lipschitz equivalence between the two structures will be the composition of the diffeomorphisms singled out in steps 1, 2, 3, 5.

By construction, the push-forward of $\mathcal{S}_1$ along a diffeomorphism ψ of M, denoted by $\psi_* \mathcal{S}_1$, is Lipschitz equivalent to $\mathcal{S}_1$ and has the same labelled graph of $\mathcal{S}_1$. Notice, moreover, that the singular locus of $\psi_* \mathcal{S}_1$ coincides with $\psi(\mathcal{Z}_1)$ and the set of tangency points coincides with $\psi(\mathcal{T}_1)$.

Step 1. Having the same labelled graph implies

$$\sum_{v \in \mathcal{V}(\mathcal{G}_1)} \operatorname{sign}(v)\chi(v) + \sum_{e \in \mathcal{E}(\mathcal{G}_1)} \tau(e) = \sum_{v \in \mathcal{V}(\mathcal{G}_2)} \operatorname{sign}(v)\chi(v) + \sum_{e \in \mathcal{E}(\mathcal{G}_2)} \tau(e).$$

By Theorem 2.1, this is equivalent to say that the Euler numbers of E_1 and E_2 are equal. Since E_1 and E_2 are oriented vector bundles of rank 2, with the same Euler number, over a compact oriented surface, then they are isomorphic. Hence, we assume $E_1 = E_2 = E$.

Step 2. Using the bijections u, k and the classification of compact oriented surfaces with boundary (see, for instance, [29]), one can prove the following lemma.

Lemma 2.15. *There exists a diffeomorphism* $\tilde{\varphi} : M \to M$ *such that* $\tilde{\varphi}(M_1^+) = M_2^+$, $\tilde{\varphi}(M_1^-) = M_2^-$, $\tilde{\varphi}|_{\mathcal{Z}_1} : \mathcal{Z}_1 \to \mathcal{Z}_2$ *is a diffeomorphism that maps* $\mathcal{T}_1$ *into* $\mathcal{T}_2$, *and, for every* $q \in \mathcal{T}_1$ $\tau^2_{\tilde{\varphi}(q)} = \tau^1_q$. *Moreover, if* $v \in \mathcal{V}(\mathcal{G}_1)$ *corresponds to* $\hat{M} \subset M \setminus \mathcal{Z}_1$, *then* $\tilde{\varphi}(\hat{M})$ *is the connected component of* $M \setminus \mathcal{Z}_2$ *corresponding to* $u(v) \in \mathcal{V}(\mathcal{G}_2)$; *if* $e \in \mathcal{E}(\mathcal{G}_1)$ *corresponds to* $W \subset \mathcal{Z}_1$, *then* $\tilde{\varphi}(W)$ *is the connected component of* $\mathcal{Z}_2$ *corresponding to* $k(e) \in \mathcal{E}(\mathcal{G}_2)$.

The lemma implies that the singular locus of $\tilde{\varphi}_* \mathcal{S}_1$ coincides with $\mathcal{Z}_2$ and the set of tangency points coincides with $\mathcal{T}_2$. For the sake of readability, in the following we rename $\tilde{\varphi}_* \mathcal{S}_1$ simply by $\mathcal{S}_1$ and we will denote by $\mathcal{Z}$ the singular locus of the two structures, by $\mathcal{T}$ the set of their tangency points, and by $M^{\pm}$ the set $M_i^{\pm}$.

Step 3. Remark that the subspaces $\Delta^1(q)$ and $\Delta^2(q)$ coincide at every ordinary and tangency point q. We are going to show that there exists a diffeomorphism of M that carries $\Delta^1(q)$ into $\Delta^2(q)$ at every point q of the manifold.

Lemma 2.16. *Let W be a connected component of $\mathcal{Z}$. There exist a tubular neighborhood $\mathbf{W}$ of W and a diffeomorphism $\varphi_W : \mathbf{W} \to \varphi_W(\mathbf{W})$ such that $d_q\varphi_W(\Delta^1(q)) = \Delta^2(\varphi_W(q))$ for every $q \in \mathbf{W}$, $\varphi_W|_W = Id|_W$ and $\varphi(\mathbf{W} \cap M^\pm) \subset M^\pm$.*

Proof. The idea of the proof is first to consider a smooth section A of $\mathrm{Hom}(TM|_W; TM|_W)$ such that for every $q \in W$, $A_q : T_qM \to T_qM$ is an isomorphism and $A_q(\Delta^1(q)) = \Delta^2(q)$. Secondly, we build a diffeomorphism φ_W of a tubular neighborhood of W such that $d_q\varphi_W = A_q$ for every point $q \in W$.

Choose on a tubular neighborhood $\mathbf{W}$ of W a parameterization (θ, t) such that $W = \{(\theta, t) \mid t = 0\}$, $M^+ \cap \mathbf{W} = \{(\theta, t) \mid t > 0\}$ and $\frac{\partial}{\partial\theta}\big|_{(\theta,0)}$ induces on W the same orientation as M^+. We are going to show the existence of two smooth functions $a, b : W \to \mathbb{R}$ such that b is positive and for every $(\theta, 0) \in W$,

$$\begin{pmatrix} 1 & a(\theta) \\ 0 & b(\theta) \end{pmatrix} (\Delta^1(\theta, 0)) = \Delta^2(\theta, 0). \tag{2.12}$$

Then, for every $q = (\theta, 0) \in W$ defining $A_q : T_qM \to T_qM$ by

$$A_{(\theta,0)} = \begin{pmatrix} 1 & a(\theta) \\ 0 & b(\theta) \end{pmatrix}, \tag{2.13}$$

we will get an isomorphism smoothly depending on the point q and carrying $\Delta^1(q)$ into $\Delta^2(q)$.

Let $W \cap \mathcal{T} = \{(\theta_1, 0), \dots, (\theta_s, 0)\}$, with $s \geq 0$. Using the chosen parametrization, there exist two smooth functions $\beta_1, \beta_2 : W \setminus \{(\theta_1, 0), \dots, (\theta_s, 0)\} \to \mathbb{R}$ such that $\Delta^i(\theta, 0) = \mathrm{span}\{(\beta_i(\theta), 1)\}$. For every $j = 1, \dots s$, there exists a smooth function g_j^i defined on a neighborhood of $(\theta_j, 0)$ in W such that $g_j^i(\theta_j) \neq 0$, $\tau_{(\theta_j,0)}^i = \mathrm{sign}(g_j^i(\theta_j))$ and

$$\beta_i(\theta) = \frac{1}{(\theta - \theta_j)g_j^i(\theta)}, \quad \theta \sim \theta_j.$$

Since the graphs associated with $\mathcal{S}_1, \mathcal{S}_2$ are equivalent, for every $j = 1 \dots s$ we have $\tau_{(\theta_j,0)}^1 = \tau_{(\theta_j,0)}^2$. Hence $\frac{g_j^2(\theta_j)}{g_j^1(\theta_j)} > 0$ for every j. Let $b : W \to \mathbb{R}$ be a positive smooth function such that for each $j \in \{1, \dots s\}$, $b(\theta_j) = \frac{g_j^2(\theta_j)}{g_j^1(\theta_j)}$. Define $a : W \to \mathbb{R}$ by

$$a(\theta) = b(\theta)\beta_2(\theta) - \beta_1(\theta).$$

Clearly a is smooth on $W \setminus \{(\theta_1, 0), \dots, (\theta_s, 0)\}$. Moreover, thanks to our choice of b, a is smooth at θ_j, and, by construction, we have (2.12). The existence of a, b is established.

Define A_q as in (2.13). Let us extend the isomorphism A_q defined for $q \in W$ to a tubular neighborhood. Define $\varphi_W : \mathbf{W} \to \mathbf{W}$ by

$$\varphi_W(\theta, t) = (a(\theta)t + \theta, b(\theta)t).$$

By construction, $d_{(\theta,0)}\varphi_W$ is an isomorphism. Hence, reducing $\mathbf{W}$ if necessary, $\varphi_W : \mathbf{W} \to \varphi_W(\mathbf{W})$ turns out to be a diffeomorphism. Finally, by definition, $\varphi_W(\theta, 0) = (\theta, 0)$ and, since b is positive, $\varphi(\mathbf{W} \cap M^\pm) \subset M^\pm$. ∎

We apply Lemma 2.16 to every connected component W of $\mathcal{Z}$. We reduce, if necessary, the tubular neighborhood $\mathbf{W}$ of W in such a way that every pair of distinct connected component of $\mathcal{Z}$ have disjoint corresponding tubular neighborhoods built as in Lemma 2.16. We claim that there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi|_{\mathbf{W}} = \varphi_W$ for every connected component W of $\mathcal{Z}$. This is a direct consequence of the fact that the labels on vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ are equal and of the classification of compact oriented surfaces with boundary (see [29]). By construction, the push-forward of $\mathcal{S}_1$ along φ is Lipschitz equivalent to $\mathcal{S}_1$ and has the same labelled graph as $\mathcal{S}_1$. To simplify notations, we denote $\varphi_* \mathcal{S}_1$ by $\mathcal{S}_1$. By Lemma 2.16, $\Delta^1(q) = \Delta^2(q)$ at every point q.

Step 4. The next point is to prove that Δ^1 and Δ^2 coincide as $\mathcal{C}^\infty(M)$-submodules.

Lemma 2.17. *The submodules Δ^1 and Δ^2 associated with $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide.*

Proof. It is sufficient to show that for every $p \in M$ there exist a neighborhood U of p such that $\Delta^1|_U$ and $\Delta^2|_U$ are generated as $\mathcal{C}^\infty(M)$-submodules by the same pair of vector fields.

If p is an ordinary point, then taking $U = M \setminus \mathcal{Z}$, we have $\Delta^1|_U = \Delta^2|_U = \mathrm{Vec}(U)$.

Let p be a Grushin point and apply Theorem 1.3 to $\mathcal{S}_1$ to find a neighborhood U of p such that

$$\Delta^1|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\}, \text{ where } F_1(x,y) = (1,0), \ F_2(x,y) = (0, xe^{\phi(x,y)}).$$

Up to reducing U we assume the existence of a frame

$$G_1(x,y) = (a_1(x,y), a_2(x,y)), \quad G_2(x,y) = (b_1(x,y), b_2(x,y))$$

such that $\Delta^2|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{G_1, G_2\}$. Since $\Delta^1(q) = \Delta^2(q)$ at every point $q \in M$, $a_2(0,y) \equiv 0$ and $b_2(0,y) \equiv 0$. Since $\Delta^2(0,y)$ is one-dimensional, let us assume $a_1(0,y) \neq 0$ for every y. Moreover, after possibly further reducing U, $\Delta^2|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{(1/a_1)G_1, G_2 - (b_1/a_1)G_1\}$ hence we may assume $a_1(x,y) \equiv 1$ and $b_1(x,y) \equiv 0$. The conditions $a_2(0,y) \equiv 0$ and $b_2(0,y) \equiv 0$ imply $a_2(x,y) = x\overline{a}_2(x,y)$ and $b_2(x,y) = x\overline{b}_2(x,y)$ respectively, with $\overline{a}_2, \overline{b}_2$ smooth functions. Since $[G_1, G_2]|_{(0,y)} = (0, \overline{b}_2(0,y))$, thanks to hypothesis (H0) on $\mathcal{S}_2$, we have $\overline{b}_2(0,y) \neq 0$. Hence, reducing U if necessary,

$$\begin{aligned}
\Delta^2|_U &= \mathrm{span}_{\mathcal{C}^\infty(M)}\{G_1 - (\overline{a}_2(x,y)/\overline{b}_2(x,y))G_2, (e^{\phi(x,y)}/\overline{b}_2(x,y))G_2\} \\
&= \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\} = \Delta^1|_U.
\end{aligned}$$

Finally, let p be a tangency point. Apply Theorem 1.3 to $\mathcal{S}_1$, i.e., choose a neighborhood U of p and a system of coordinates (x,y) such that $p = (0,0)$,

$$\Delta^1|_U = \mathrm{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\}, \text{ where } F_1(x,y) = (1,0), \ F_2(x,y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}),$$

and ψ, ξ are smooth functions such that $\psi(0) > 0$. Consider the change of coordinates

$$\tilde{x} = x, \quad \tilde{y} = y - x^2\psi(x).$$

Then
$$F_1(\tilde{x}, \tilde{y}) = (1, \tilde{x}a(\tilde{x})), \quad F_2(\tilde{x}, \tilde{y}) = (0, \tilde{y}e^{\xi(\tilde{x}, \tilde{y} + \tilde{x}^2\psi(\tilde{x}))}),$$

where $a(\tilde{x}) = -2\psi(\tilde{x}) - \tilde{x}\psi'(\tilde{x})$. To simplify notations, in the following we rename $\tilde{x}, \tilde{y}$ by x, y respectively and we still denote by $\xi(x, y)$ the function $\xi(x, y + x^2\psi(x))$. In the new coordinate system we have $p = (0, 0)$, $\mathcal{Z} \cap U = \{(x, y) \mid y = 0\}$, $F_1(x, y) = (1, xa(x))$ and $F_2(x, y) = (0, ye^{\xi(x,y)})$. Reducing U, if necessary, let $G_1(x, y) = (a_1(x, y), a_2(x, y)), G_2(x, y) = (b_1(x, y), b_2(x, y))$ be a frame for $\Delta^2|_U$. Since $\Delta^1(q) = \Delta^2(q)$ at every point, we have $a_2(0, 0) = b_2(0, 0) = 0$. Since $\Delta^2(0, 0)$ is one-dimensional, we may assume $a_1(0, 0) \neq 0$. After possibly further reducing U, $\Delta^2|_U = \text{span}_{\mathcal{C}^\infty(M)}\{(1/a_1)G_1, G_2 - (b_1/a_1)G_1\}$ and we can assume $a_1(x, y) \equiv 1$ and $b_1(x, y) \equiv 0$. Moreover, by $\Delta^1(x, 0) = \Delta^2(x, 0)$ we get $a_2(x, 0) = xa(x)$ and $b_2(x, 0) \equiv 0$, whence $a_2(x, y) = xa(x) + y\overline{a}_2(x, y)$ and $b_2(x, y) = y\overline{b}_2(x, y)$, with $\overline{a}_2, \overline{b}_2$ smooth functions. Computing the Lie brackets we get

$$[G_1, G_2]|_{(x,0)} = (0, xa\overline{b}_2)|_{(x,0)}, \quad [G_1, [G_1, G_2]]|_{(0,0)} = (0, a\overline{b}_2))|_{(0,0)}.$$

Applying hypothesis (H0) to $\mathcal{S}_2$ we have $\overline{b}_2(x, 0) \neq 0$ for all x in a neighborhood of 0. Hence, up to reducing U,

$$
\begin{aligned}
\Delta^2|_U &= \text{span}_{\mathcal{C}^\infty(M)}\{G_1 - (\overline{a}_2(x, y)/\overline{b}_2(x, y))G_2, (e^{\xi(x,y)}/\overline{b}_2(x, y))G_2\} \\
&= \text{span}_{\mathcal{C}^\infty(M)}\{F_1, F_2\} = \Delta^1|_U.
\end{aligned}
$$

∎

Step 5. Thanks to Lemma 2.17 and Proposition 2.10 we can assume $f_1 = f_2 = f$. In other words, we reduce to the case $\mathcal{S}_1 = (E, f, \langle \cdot, \cdot \rangle_1)$ and $\mathcal{S}_2 = (E, f, \langle \cdot, \cdot \rangle_2)$. By compactness of M, there exists a constant $k \geq 1$ such that

$$\frac{1}{k}\langle u, u \rangle_2 \leq \langle u, u \rangle_1 \leq k\langle u, u \rangle_2, \ \forall u \in E. \tag{2.14}$$

For every $q \in M$ and $v \in \Delta(q)$ let $\mathbf{G}_q^i(v) = \inf\{\langle u, u \rangle_i \mid u \in E_q, f(u) = v\}$ (see Chapter 1). Clearly,

$$\frac{1}{k}\mathbf{G}_q^2(v) \leq \mathbf{G}_q^1(v) \leq k\mathbf{G}_q^2(v), \ \forall v \in f(E_q). \tag{2.15}$$

By (2.14), admissible curves for $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide. Moreover, given an admissible curve $\gamma : [0, T] \to M$, we can compare its length with respect to $\mathcal{S}_1$ and $\mathcal{S}_2$ using (2.15). Namely,

$$\frac{1}{\sqrt{k}}\int_0^T \sqrt{\mathbf{G}_{\gamma(s)}^2(\dot{\gamma}(s))}ds \leq \int_0^T \sqrt{\mathbf{G}_{\gamma(s)}^1(\dot{\gamma}(s))}ds \leq \sqrt{k}\int_0^T \sqrt{\mathbf{G}_{\gamma(s)}^2(\dot{\gamma}(s))}ds.$$

Since the Carnot-Caratheodory distance between two points is defined as the infimum of the lengths of the admissible curves joining them, we get

$$\frac{1}{\sqrt{k}}d_2(p, q) \leq d_1(p, q) \leq \sqrt{k}d_2(p, q), \ \forall p, q \in M.$$

This is equivalent to say that the identity map is a Lipschitz equivalence between $\mathcal{S}_1$ and $\mathcal{S}_2$. ∎

Local analysis at tangency points

This chapter is mainly devoted to the local analysis of 2-ARSs at tangency points.

Such points are the most difficult to handle because of the fact that the asymptotic of the distance is different from the two sides of the singular set. An open question is how to find a local representation for the orthonormal frame at tangency points which is completely reduced, in the sense that it depends only on the 2-ARS. In this chapter we address to this problem starting from the construction of local representations given in Theorem 1.3. In order to build the coordinate system for which these local expressions apply, the idea is the following. Consider a smooth parametrized curve passing through a point q. If the curve is assumed to be transversal to the distribution at each point, then the Carnot–Caratheodory distance from the curve is shown to be smooth on a neighborhood of q. Given a point p near q the first coordinate of p is, by definition, the distance between p and the chosen curve, with a suitable choice of sign. The second coordinate of p is the parameter corresponding to the point on the chosen curve that realizes the distance between p and the curve. If the parameterized curve used in this construction can be built canonically, then one gets a local representation that cannot be further reduced. For Riemannian points, a canonical parametrized curve transversal to the distribution can be easily identified, at least in the generic case (see Proposition 3.1). For Grushin points, a canonical curve transversal to the distribution is the singular set which has also a natural parameterization as explained in Section 3.1.

As concerns tangency points, in Theorem 1.3 the choice of the smooth parameterized curve is arbitrary and not canonical. Here we find a canonical one. As a first candidate, we consider the cut locus from the tangency point. Using the nilpotent approximation, we study the wave front from the tangency point (Proposition 3.7). This allows us to estimate both the conjugate locus (Proposition 3.8) and the cut locus. It turns out that in general the cut locus from the tangency point is not smooth but has an asymmetric cusp (see Proposition 3.2), whence it is not a good choice for our purpose. Another possible candidate is the cut locus from the singular set in a neighborhood of the tangency point. Nevertheless, we prove that in general this curve is non-smooth (Proposition 3.3) and accumulates at the tangency point transversally to the distribution on one side of the singular set, tangentially on the other side. A third possibility is to look for crests or valleys of the Gaussian curvature which intersect transversally the singular set at a tangency point. The main result (Theorem 3.4) consists in the proof of the existence of such a crest. Moreover, this curve admits a canonical regular parameterization. A completely reduced local representation is then obtained implicitly by requiring this curve to be the vertical axis. Explicit relations between the Taylor coefficients of the functions appearing in

the local representation can be further obtained (Proposition 3.5).

The structure of the chapter is the following. In Section 3.1 we state the main results concerning the shape of the cut loci from a tangency point and from the singular set and the crest of the curvature passing through the tangency point. A local representation for Riemannian points is built in Section 3.2. Then, in Section 3.3.1 we start the local analysis at tangency points by computing the jet of the exponential map. This allows us to estimate in Section 3.3.2 the conjugate locus and in Section 3.3.3 the cut locus from a tangency point. Section 3.3.4 is devoted to proving that the cut locus from the singular set is not smooth. Finally, we end this chapter showing the existence of a crest of the curvature passing transversally to the distribution at a tangency point.

All the results in Section 3.1 except Proposition 3.2 are to be found in [15]. Propositions 3.2, 3.7 and 3.8 appear in [11].

3.1   Completely reduced local representations and statement of the main results

Definition 3.1. We say that a local representation is *completely reduced* if it corresponds to a canonical choice in Definition 1.6 of φ, F_1 and F_2, up to orientation.

Remark 3.1. The expression "up to orientation" is necessary since, if $\varphi = (\varphi_1, \varphi_2)$, there is no way in general to choose canonically among (φ_1, φ_2), $(-\varphi_1, \varphi_2)$, $(-\varphi_1, -\varphi_2)$, $(\varphi_1, -\varphi_2)$, and among (F_1, F_2), $(-F_1, F_2)$, $(-F_1, -F_2)$, $(F_1, -F_2)$.

Remark 3.2. Here by canonical we mean that it depends only on the 2-ARS. In this case if $X = (X_1(x,y), X_2(x,y))$ and $Y = (Y_1(x,y), Y_2(x,y))$ then up to sign and up to the transformations $x \to -x$ and $y \to -y$, we have that $X_1(x,y)$, $X_2(x,y)$, $Y_1(x,y)$, $Y_2(x,y)$ are functional invariants of the system.

The construction in the proof of Lemma 1.4 provides a coordinate system such that the corresponding local representation (X, Y) at a point q has the form $X = \partial_x$ and $Y = a(x,y)\partial_y$. Conversely, if a local representation of the form $X = \partial_x$, $Y = a(x,y)\partial_y$ is given, then the vertical axis $W = \{(x,y) \mid x = 0\}$ is transversal to the distribution and the distance of a point (x,y) from W is $|x|$. Hence we have the following:

Claim: up to orientation, constructing a local representation of the form $X = \partial_x$, $Y = a(x,y)\partial_y$ is equivalent to choosing a parameterized curve transversal to the distribution.

Thanks to the claim, constructing a completely reduced local representation of the type $(\partial_x, a(x,y)\partial_y)$ is equivalent to choosing a canonical parameterized curve transversal to the distribution.

Among the local representations given in Theorem 1.3, (F2) is completely reduced. Indeed, in the proof of Theorem 1.3 we choose as curve W the singular set $\mathcal{Z}$, which is naturally associated to the structure. It is easy to see that for any orthonormal frame (G_1, G_2), the Lie bracket $[G_1, G_2]|_{\mathcal{Z}}$ modulo elements in Δ does

not change. As for the parametrization of $\mathcal{Z}$, the choice in Theorem 1.3 is such that $[F_1, F_2]|_{\mathcal{Z}} = \frac{\partial}{\partial y}$ modulo Δ. For what concerns (F1) and (F3), they are not completely reduced since the curve transversal to the distribution is chosen arbitrarily. Our aim is to provide at Riemannian and tangency points a canonical choice of a parametrized curve associated with the structure.

First, let us consider the case of Riemannian points. Generically, the set of Riemannian points $p \in M$ such that the gradient of the Gaussian curvature K is singular is a discrete set Π and at each point of Π exactly one crest and one valley of K pass through the point. Hence at a point outside Π, one can choose as W the level set of the curvature, parameterized by arclength. For points of Π, one can choose the crest or the valley parameterized by arclength. In the following proposition we sum up the analysis of local representations at Riemannian points. For the sake of readability, the proof is postponed to Section 3.2.

Proposition 3.1. *Let $q \in M$ be a Riemannian point of a generic 2-ARS. If $\nabla K(q) \neq 0$, then a completely reduced local representation for $\mathcal{S}$ at q is (F1) where $\phi(0, y) \equiv 0$ and*

$$-2\partial_x^2 \phi(0, y)\partial_x\partial_y\phi(0, y) + \partial_x^2\partial_y\phi(0, y) \equiv 0.$$

If $\nabla K(q) = 0$, then a local representation for $\mathcal{S}$ at q is (F1) where $\phi(0, y) \equiv 0$ and $h_0 = 0$ (h_0 defined below in formula (3.3)).

The case of tangency points is rather complicated. The first candidate as smooth curve is the cut locus from the tangency point. The shape of this cut locus (see Figure 3.1(a)) can be computed as the following result states.

Proposition 3.2. *Let $\mathcal{S}$ be a 2-ARS on M satisfying (H0). Let $q \in M$ be a tangency point such that there exists a local representation of the type (F3) for $\mathcal{S}$ at q with the property*

$$\psi'(0) + \psi(0)\partial_x\xi(0, 0) \neq 0.$$

Then the cut locus from the tangency point accumulates at q as an asymmetric cusp whose branches are separated locally by $\mathcal{Z}$. In the coordinate system where the chosen local representation is (F3), the cut locus accumulates as the set

$$\{(x, y) \mid y > 0, y^2 - \alpha_1 x^3 = 0\} \cup \{(x, y) \mid y < 0, y^2 - \alpha_2 x^3 = 0\},$$

with $\alpha_i = c_i e^{2\xi(0,0)}/(\psi'(0) + \psi(0)\partial_x\xi(0, 0))^3$, the constants c_i being nonzero and independent on the structure.

The proof of this result is given in Section 3.3.3. Due to Proposition 3.2, in general the cut locus is not smooth and cannot be used to find a completely reduced local representation. Another candidate would be the cut locus from $\mathcal{Z}$ in a neighborhood of a tangency point. A description of such locus is given by the following proposition (see also Figure 3.1(b)).

Proposition 3.3. *Let $\mathcal{S}$ be a 2-ARS on M satisfying (H0). Let $q \in M$ be a tangency point such that there exists a local representation of the type (F3) for $\mathcal{S}$ at q with the property*

$$\psi'(0) + \psi(0)\partial_x\xi(0, 0) \neq 0.$$

(a) the cut locus from the tangency point

(b) the cut locus from the singular set

(c) crests of the curvature
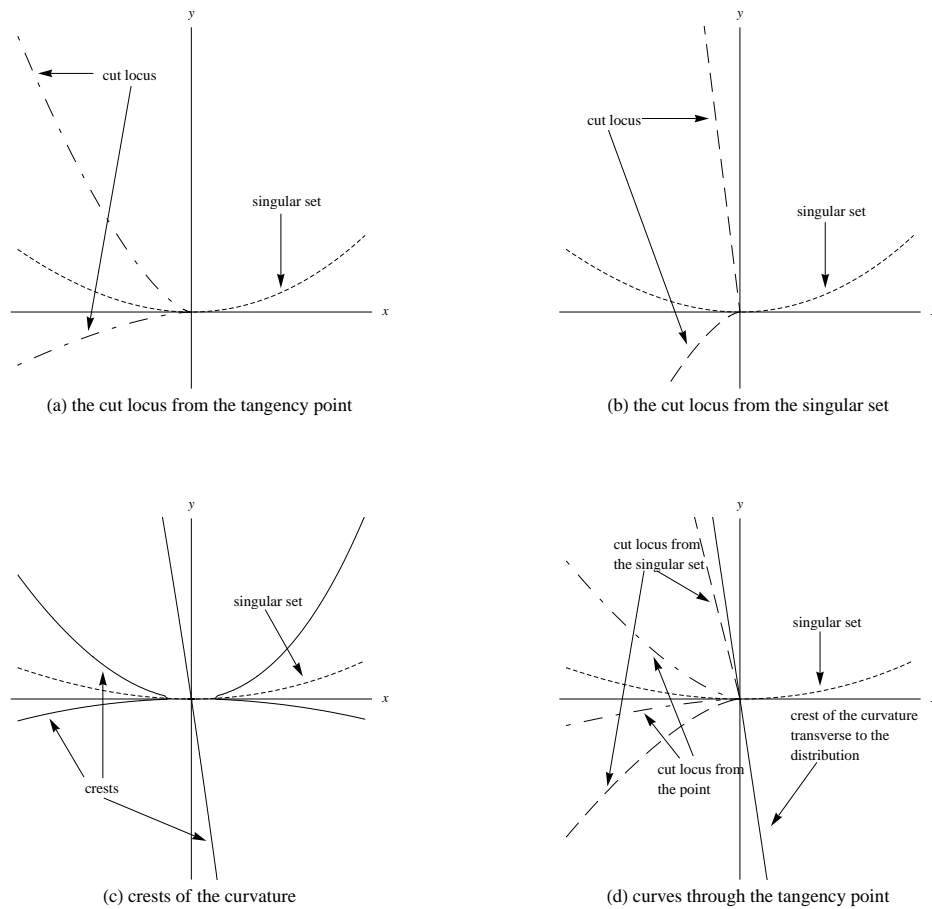
(d) curves through the tangency point

Figure 3.1: The singular locus (dotted line), the cut locus from a tangency point (semidashed line), the cut locus from the singular set (dashed line), and the crests of the Gaussian curvature (solid lines) for the ARS with orthonormal frame $F_1 = \frac{\partial}{\partial x}$, $F_2 = (y - x^2 - x^3)\frac{\partial}{\partial y}$. In this case there are three crests of the curvature. Notice that all the crests except only one are tangent to the distribution.

Then the cut locus from the singular set $\mathcal{Z}$ in a neighborhood of q accumulates at q as the union of two curves locally separated by $\mathcal{Z}$, one converging to q transversally to $\mathcal{Z}$, the other one with tangent direction at q belonging to the distribution. In the chosen local representation, the tangent line at q to the part of the cut locus which is transverse to the distribution is $x = -\frac{1}{2}\psi'(0)y$.

The proof of Proposition 3.3 is in the same spirit of the proof of Proposition 3.2 and it is postponed to Section 3.3.4. Notice that, being non-smooth at the tangency point, the cut locus from $\mathcal{Z}$ cannot be chosen for our purpose.

Finally, we look for a candidate curve to build a completely reduced local representation among the crests or valleys of the curvature. Generically, a crest passing through a tangency point, being smooth and transverse to the distribution, happens to exist and to be unique. Moreover, along this curve the scalar product between

the tangent vector to the curve and the gradient of the curvature is smooth and nonvanishing when prolonged to the tangency point. Requiring it to be identically equal to 1, we fix a canonical parameterization. More precisely we get the following result.

Theorem 3.4. *Let q be a tangency point of a 2-ARS satisfying (H0). Then there exists $\epsilon > 0$ and a unique smooth parametrized curve Γ defined on $(-\epsilon, \epsilon)$ which satisfies the following properties: (i) $\Gamma(0) = q$, $\Gamma'(0) \notin \Delta(q)$; (ii) the support of Γ is contained in a crest of the Gaussian curvature K; (iii) $G(\Gamma'(t), \nabla K(\Gamma(t))) \equiv 1$.*

Notice that crests and valleys of the curvature are included in the set of points such that

$$G(\nabla(||\nabla K||^2), \nabla K^{\perp}) = 0\}, \tag{3.1}$$

where ∇K is the almost-Riemannian gradient of K, i.e., the unique vector such that $G(\nabla K, \cdot) = dK(\cdot)$, $||\nabla K||^2 = G(\nabla K, \nabla K)$, and $(\nabla K)^{\perp}$ satisfies $G(\nabla K, (\nabla K)^{\perp}) = 0$ (see Figure 3.1(c)).

The curve given by Theorem 3.4 can be used to reduce completely the local representation (F3). Unfortunately, since in the proof of Theorem 3.4 the crest is obtained as an implicit solution of equation (3.1), we cannot get explicitly the relations between the functions ψ and ξ. However one can compute relations among their Taylor coefficients at $(0, 0)$. For instance at the first order we get the following result.

Proposition 3.5. *In the local representation (F3) we can choose the functions ξ, ψ such that $2\xi_x(0,0)\psi(0) - 3\psi'(0) = 0$.*

The proof of Theorem 3.4 is given in Section 3.3.5.

3.2 Riemannian points

Proof of Proposition 3.1. Let $q \in M$ be such that $\nabla K(q) \neq 0$. In this case, the level set $\{p \in M \mid K(p) = K(q)\} \cap U$ is a smooth 1-dimensional submanifold of M. Using the local representation (F1), one gets

$$\nabla K(x,y) = (\partial_x^3 \phi(x,y) - 2\partial_x \phi(x,y)\partial_x^2 \phi(x,y), e^{2\phi(x,y)}\left(-2\partial_x \phi(x,y)\partial_x \partial_y \phi(x,y) + \partial_x^2 \partial_y \phi(x,y)\right)).$$

Requiring that the level set of the curvature passing through q is the vertical axis, one gets that the second coordinate of $\nabla K(0, y)$ is zero. Hence we get

$$-2\partial_x \phi(0,y)\partial_x \partial_y \phi(0,y) + \partial_x^2 \partial_y \phi(0,y) \equiv 0.$$

Requiring that the vertical axis is parameterized by arclength, one gets

$$\phi(0, y) \equiv 0.$$

Assume now that q is such that $\nabla K(q) = 0$. The crests or valleys of K are implicitly defined by the equation

$$G(\nabla(||\nabla K||^2), (\nabla K)^{\perp}) = 0. \tag{3.2}$$

Consider an orthonormal frame of the type $(F1)$. The jet of the left hand side of equation (3.2) is

$$h(x,y) = h_0 x^2 + h_1 xy - h_0 y^2 + \sum_{i=0}^{3} c_i(x,y) x^i y^{3-i},$$

where c_i are smooth functions and h_i are real numbers depending on the values of ϕ and its derivatives until order 4 at $(0,0)$. We study the generic case $(h_0, h_1) \neq (0,0)$. In this case h has a saddle in $(0,0)$ and the equation $h(x,y) = 0$ defines locally two smooth curves which are respectively the crest and the valley of the curvature. Requiring that the vertical axis is a crest or a valley, we have $h_0 = 0$. Finally one can parameterize the crest by arclength. The explicit expression of h_0 is

$$
\begin{aligned}
h_0 &= 2e^{4\phi} \, (\, 4\partial_x^2 \partial_y^1 \phi^3 \partial_x^3 \phi - 4\partial_x^1 \partial_y^2 \phi \partial_x^2 \partial_y \phi \partial_x^3 \phi^2 + 8\partial_x \phi^3 (\partial_x^2 \partial_y \phi^3 - \partial_x \partial_y^2 \phi \partial_x^2 \partial_y \phi \partial_x^3 \phi) - \\
&\quad - \; 2\partial_x^2 \phi^2 \partial_x^2 \partial_y^2 \phi \partial_x^3 \partial_y \phi + 2\partial_x \partial_y^2 \phi \partial_x^2 \phi \partial_x^3 \phi \partial_x^3 \partial_y \phi - \partial_x^3 \partial_y \phi^3 + \partial_x^2 \partial_y^2 \phi \partial_x^3 \partial_y \phi \partial_x^4 \phi - \\
&\quad - \; 2\partial_x \partial_y \phi^2 \partial_x^3 \partial_y \phi (4\partial_x^2 \phi^2 + \partial_x^4 \phi) + 4\partial_x \partial_y \phi^3 (\partial_x^3 \phi^2 + \partial_x^2 \phi \partial_x^4 \phi) + \\
&\quad + \; 2\partial_x \partial_y \phi(2\partial_x^2 \phi^3 \partial_x^2 \partial_y^2 \phi - 2\partial_x \partial_y^2 \phi \partial_x^2 \phi^2 \partial_x^3 \phi + 2\partial_x^2 \partial_y^2 \phi \partial_x^3 \phi^2 - 5\partial_x^2 \partial_y \phi \partial_x^3 \phi \partial_x^3 \partial_y \phi + \\
&\quad + \; 3\partial_x^2 \phi \partial_x^3 \partial_y \phi^2 + (\partial_x^2 \partial_y \phi^2 - \partial_x^2 \phi \partial_x^2 \partial_y^2 \phi + \partial_x \partial_y^2 \phi \partial_x^3 \partial_y \phi)\partial_x^4 \phi) + \qquad (3.3) \\
&\quad + \; 4\partial_x \phi^2(-3\partial_x^2 \partial_y \phi^2 \partial_x^3 \partial_y \phi + \partial_x \partial_y^2 \phi \partial_x^3 \phi \partial_x^3 \partial_y \phi + \partial_x^2 \partial_y \phi(\partial_x^2 \partial_y^2 \phi \partial_x^3 \phi + \partial_x \partial_y^2 \phi \partial_x^4 \phi)) + \\
&\quad + \; \partial_x \phi(2\partial_x^3 \partial_y \phi(3\partial_x^2 \partial_y \phi \partial_x^3 \partial_y \phi - \partial_x \partial_y^2 \phi \partial_x^4 \phi) - 2\partial_x^2 \partial_y^2 \phi(\partial_x^3 \phi \partial_x^3 \partial_y \phi + \partial_x^2 \partial_y \phi \partial_x^4 \phi)) \,) \, ,
\end{aligned}
$$

where all the derivatives of ϕ are computed at $(0,0)$.   ∎

3.3   Tangency points

3.3.1   Jet of the exponential map at a tangency point

Let us analyse an ARS satisfying hypothesis (H0) in a neighborhood of a tangency point, that is, consider the almost-Riemannian structure on $\mathbb{R}^2$ whose orthonormal frame is

$$F_1(x,y) = (1,0), \quad F_2(x,y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}), \qquad (3.4)$$

where $\psi(0) > 0$. Taking the coordinate change $\tilde{x} = x, \tilde{y} = y/\psi(0)$, in the orthonormal frame (3.4) we may assume $\psi(0) = 1$. For the sake of readability, throughout this section we rename $\tilde{x}, \tilde{y}$ by x, y.

The coordinates (x,y) have weights respectively $(1,3)$ and they are privileged at $(0,0)$ (for the definition of privileged coordinates and nilpotent approximation see, for instance, [9]). Hence, the nilpotent approximation of (3.4) at $(0,0)$ is the ARS defined by the orthonormal frame

$$\hat{F}_1(x,y) = (1,0), \quad \hat{F}_2(x,y) = (0, -\gamma x^2), \qquad (3.5)$$

where $\gamma = e^{\xi(0,0)}$. The singular locus of this structure is the y-axis and at each singular point the distribution is transverse to the singular locus, while the growth vector is $(1,1,2)$. Hence, this ARS does not satisfy the generic assumption (H0). For

this reason, the optimal synthesis starting at $(0,0)$ for the nilpotent approximation does not present the same properties as the optimal synthesis at a tangency point for the generic case. Nevertheless, such an optimal synthesis can be explicitly computed (see [2]).

Proposition 3.6. *Consider the almost-Riemannian structure whose orthonormal frame is $\hat{F}_1, \hat{F}_2$. Then, denoting by $(x(t,a), y(t,a))$ the geodesic starting at $(0,0)$ with initial covector $(\pm 1, a)$ we have $x(t,0) = \pm t$, $y(t,0) = 0$ and for $a \neq 0$*

$$
x(t,a) = \mp \frac{1}{\sqrt{\gamma|a|}} \mathrm{cn}\,(\mathcal{K} + t\sqrt{2\gamma|a|})
$$

$$
y(t,a) = \frac{\mathrm{sign}(a)}{3\sqrt{2\gamma|a|^{3/2}}}[t\sqrt{2\gamma|a|} + 2\mathrm{sn}\,(\mathcal{K} + t\sqrt{2\gamma|a|})\mathrm{cn}\,(\mathcal{K} + t\sqrt{2\gamma|a|})\mathrm{dn}\,(\mathcal{K} + t\sqrt{2\gamma|a|})],
$$

where $\mathcal{K}$ is the complete elliptic integral of the first kind

$$
\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - 1/2 \sin^2 \varphi}}
$$

and $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$ denote the Jacobi elliptic functions.

The cut instant along a geodesic with $a \neq 0$ is $t_{cut} = \sqrt{2}\mathcal{K}/\sqrt{\gamma|a|}$, whence the cut locus is the set $\{(x,y) \mid x = 0\} \setminus \{(0,0)\}$. The first conjugate instant along a geodesic with $a \neq 0$ is $t_{conj} \sim 3\mathcal{K}/\sqrt{2\gamma|a|}$ and the conjugate locus accumulates at the origin as a set of the form $\{(x,y) \mid y = \alpha x^3\} \cup \{(x,y) \mid y = -\alpha x^3\} \setminus \{(0,0)\}$, with $\alpha \neq 0$.

For the proof of Proposition 3.6 see, for instance [2].

Our aim is to use the optimal synthesis for the nilpotent approximation to compute the jet of the exponential map at the origin for the ARS in (3.4). The development at $(0,0)$ of the orthonormal frame in (3.4) truncated at order zero is

$$
\tilde{F}_1 = \frac{\partial}{\partial x}, \tilde{F}_2 = \gamma(y - x^2 - \varepsilon' x^3)\frac{\partial}{\partial y}, \tag{3.6}
$$

where $\varepsilon' = \psi'(0) + \xi_x(0,0)$. The following proposition computes the exponential map at the origin for the ARS defined in (3.6). As we shall see in the proof of this proposition, the higher order terms in the expansion of the elements of the orthonormal frame in (3.4) do not affect the estimation of the exponential map and, consequently, the order zero is sufficient to describe the cut and conjugate loci from the tangency point, at least qualitatively.

Proposition 3.7. *Consider the ARS on $\mathbb{R}^2$ defined by the orthonormal frame given in (3.6). The solution of the Hamiltonian system associated with*

$$
H(q,p) = \frac{1}{2}(p_x^2 + \gamma^2(y - x^2 - \varepsilon' x^3)^2 p_y^2)
$$

with initial condition $(x, y, p_x, p_y)|_{t=0} = (0, 0, \pm 1, a)$ with $|a| \sim +\infty$ can be expanded as

$$
x(t) = \eta X^0(t/\eta) + \eta^2 X^1(t/\eta) + o(\eta^2),
$$
$$
y(t) = \eta^3 Y^0(t/\eta) + \eta^4 Y^1(t/\eta) + o(\eta^4),
$$

where $\eta = \frac{1}{\sqrt{|a|}}$,

$$\begin{cases} \dot{X}^0 & = & P_X^0 \\ \dot{Y}^0 & = & \gamma^2 P_Y^0 (X^0)^4 \\ \dot{P}_X^0 & = & -2\gamma^2 X^{0\,3} P_Y^{0\,2} \\ \dot{P}_Y^0 & = & 0 \end{cases} \tag{3.7}$$

with initial condition $(X^0, Y^0, P_X^0, P_Y^0)|_{t=0} = (0, 0, \pm 1, \text{sign}(a))$ and

$$\begin{cases} \dot{X}^1 & = & P_X^1, \\ \dot{Y}^1 & = & \gamma^2 (P_Y^1 (X^0)^4 + 4 P_Y^0 (X^0)^3 X^1 - 2 P_Y^0 ((X^0)^2 Y^0 - \varepsilon'(X^0)^5)), \\ \dot{P}_X^1 & = & -\gamma^2 (4 P_Y^0 P_Y^1 (X^0)^3 + 6 (P_Y^0)^2 (X^0)^2 X^1 - (P_Y^0)^2 (2 X^0 Y^0 - 5\varepsilon'(X^0)^4)) \\ \dot{P}_Y^1 & = & \gamma^2 P_Y^{0\,2} X^{0\,2}, \end{cases} \tag{3.8}$$

with initial condition $(X^1, Y^1, P_X^1, P_Y^1)|_{t=0} = (0, 0, 0, 0)$.

Proof. The Hamiltonian system associated with H is

$$\begin{cases} \dot{x} & = & p_x \\ \dot{y} & = & \gamma^2 (y - x^2 - \varepsilon' x^3)^2 p_y \\ \dot{p}_x & = & -\gamma^2 p_y^2 (y - x^2 - \varepsilon' x^3)(-2x - 3\varepsilon' x^2) \\ \dot{p}_y & = & -\gamma^2 p_y^2 (y - x^2 - \varepsilon' x^3). \end{cases}$$

According to the weights we set

$$\begin{array}{rclcrcl} x & = & \eta X, & p_x & = & P_X, \\ y & = & \eta^3 Y, & p_y & = & \frac{P_Y}{\eta^2}, \end{array}$$

where η is a parameter. The evolution equations for (X, Y, P_X, P_Y) are

$$\begin{cases} \dot{X} & = & \frac{P_X}{\eta} \\ \dot{Y} & = & \gamma^2 P_Y (\frac{X^4}{\eta} - 2 X^2 Y + 2\varepsilon' X^5 + \eta (Y^2 + \varepsilon'^2 X^6 - 2\varepsilon' X^3 Y)) \\ \dot{P}_X & = & -\gamma^2 P_Y^2 \left(2\frac{X^3}{\eta} - 2 X Y + 5\varepsilon' X^4 - 3\varepsilon' \eta X^2 (Y - \varepsilon' X^3) \right) \\ \dot{P}_Y & = & -\gamma^2 P_Y^2 (-X^2 + \eta (Y - \varepsilon' X^3)). \end{cases}$$

Defining the new time parameter $s = t/\eta$, the evolution equations become

$$\begin{cases} \frac{dX}{ds} & = & P_X \\ \frac{dY}{ds} & = & \gamma^2 P_Y (X^4 - 2\eta (X^2 Y - \varepsilon' X^5) + \eta^2 (Y^2 + \varepsilon'^2 X^6 - 2\varepsilon' X^3 Y)) \\ \frac{dP_X}{ds} & = & -\gamma^2 P_Y^2 \left(2 X^3 - \eta (2 X Y - 5\varepsilon' X^4) - 3\varepsilon' \eta^2 X^2 (Y - \varepsilon' X^3) \right) \\ \frac{dP_Y}{ds} & = & -\gamma^2 P_Y^2 (-\eta X^2 + \eta^2 (Y - \varepsilon' X^3)). \end{cases} \tag{3.9}$$

Consider the expansions with respect to η

$$\begin{array}{rclcrcl} X & = & X^0 + \eta X^1 + o(\eta), & P_X & = & P_X^0 + \eta P_X^1 + o(\eta), \\ Y & = & Y^0 + \eta Y^1 + o(\eta), & P_Y & = & P_Y^0 + \eta P_Y^1 + o(\eta). \end{array}$$

By identification the leading terms satisfy

$$\begin{cases} \frac{dX^0}{ds} &= P_X^0 \\ \frac{dY^0}{ds} &= \gamma^2 P_Y^0 (X^0)^4 \\ \frac{dP_X^0}{ds} &= -2\gamma^2 (P_Y^0)^2 (X^0)^3 \\ \frac{dP_Y^0}{ds} &= 0. \end{cases} \tag{3.10}$$

In particular P_Y^0 is constant. For $p_y(0) = a \neq 0$ we can fix $\eta = 1/\sqrt{|a|}$ and then P_Y^0 is normalized to $\mathrm{sign}(a)$. System (3.10) coincides with the Hamiltonian system for the nilpotent model. Using Proposition 3.6, the solution to (3.10) with initial condition $(0, 0, P_X^0(0), P_Y^0(0))$ is

$$\begin{cases} X^0(s) &= -\frac{P_X^0(0)}{\sqrt{\gamma}} \mathrm{cn}\,(\mathcal{K} + s\sqrt{2\gamma}) \\ Y^0(s) &= \frac{P_Y^0(0)}{3\sqrt{2\gamma}}(s\sqrt{2\gamma} + 2\mathrm{sn}\,(\mathcal{K} + s\sqrt{2\gamma})\mathrm{cn}\,(\mathcal{K} + s\sqrt{2\gamma})\mathrm{dn}\,(\mathcal{K} + s\sqrt{2\gamma}) \\ P_X^0(s) &= \sqrt{2}P_X^0(0)\mathrm{dn}\,(\mathcal{K} + s\sqrt{2\gamma})\mathrm{sn}\,(\mathcal{K} + s\sqrt{2\gamma}) \\ P_Y^0(s) &\equiv P_Y^0(0) = \mathrm{sign}\,a. \end{cases} \tag{3.11}$$

Identifying terms of order 1 in (3.9) we get system (3.8). ∎

3.3.2 The conjugate locus at a tangency point

The following result gives a description of the conjugate locus at a tangency point of a generic 2-ARS.

Proposition 3.8. *Consider an ARS on $\mathbb{R}^2$ defined by the orthonormal frame (3.4). Then there exists a constant $\alpha \neq 0$ such that the conjugate locus from $(0, 0)$ accumulates at $(0, 0)$ as the set*

$$\{(x, y) \mid y = \alpha x^3\} \cup \{(x, y) \mid y = -\alpha x^3\} \setminus \{(0, 0)\}.$$

Proof. Applying Proposition 3.7, the exponential map at $(0, 0)$ is given by

$$(\eta, s) \mapsto (\eta X^0(s) + o(\eta), \eta^3 Y^0(s) + o(\eta^3)),$$

where $s = t\sqrt{|p_y(0)|}$, η parametrizes the initial covector as $(p_x(0) = \pm 1, p_y(0) = P_Y^0/\eta^2)$, and X^0, Y^0 are given in (3.11). In order to compute the conjugate time, we look for the first zero of the Jacobian of the exponential map. The Jacobian is equal, up to a multiplicative constant, to

$$\eta^3 \left(X^0 \frac{dY^0}{ds} - 3Y^0 \frac{dX^0}{ds}\right) + o(\eta^3) = \eta^3 v(s) + o(\eta^3).$$

It was proved in [12] that v has its first positive zero at $s = s_0 \sim 3\mathcal{K}/\sqrt{2\gamma}$ and that $v'(s_0) \neq 0$. Hence, the conjugate time is of the form $s_0 + o(1)$ where $o(1)$ is a continuous function going to zero when η goes to zero.

The exponential map for the nilpotent approximation has only stable singularities (folds) corresponding to the first conjugate locus. Thus, considering the exponential map of the ARS defined by (3.4) as a small deformation of the nilpotent case, the first conjugate locus at $(0, 0)$ for the generic case accumulates at $(0, 0)$ as a set of the form $\{(x, y) \mid (y - \alpha x^3)(y + \alpha x^3) = 0\}$, where $\alpha \neq 0$, see Proposition 3.6. ∎

3.3.3 The cut locus at a tangency point

In this section we show Proposition 3.2. As one can infer from the following proof, the shape of the cut locus is determined only by the terms of order up to zero in the expansion of the elements of the orthonormal frame. Higher order terms do not contribute to the estimation of the way the cut locus approaches to the tangency point. Figure 3.2 portraits spheres centered at the tangency point with small radius for three different structures, showing that the symmetry of the nilpotent approximation is lost in general.
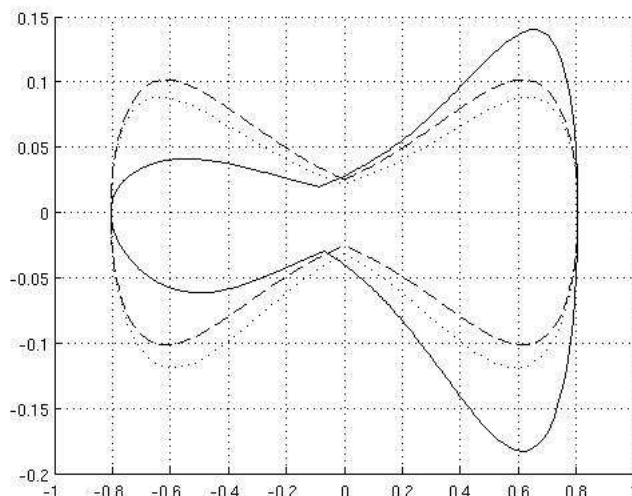

Figure 3.2: The spheres of small radius for the nilpotent approximation (dotted line) and for an ARS as in (3.6) with $\varepsilon' = 0$ (dashed line) are symmetric. They are not C^1 at their intersection with the cut locus, which in both cases is the vertical axis. The sphere of small radius for an ARS as in (3.6) with $\varepsilon' \neq 0$ (solid line) loses the symmetry

Proof of Proposition 3.2. Consider the ARS on $\mathbb{R}^2$ defined in (3.4). In the following, we restrict our analysis to the branch of the cut locus at $(0,0)$ in the upper half-plane $\{(x,y) \mid y > 0\}$, the computations being analogous for the branch contained in the lower half-plane.

Let (X^0, Y^0, P_X^0, P_Y^0) be as in (3.11) with initial condition condition $(0,0,1,1)$. Moreover, denote by (X^1, Y^1, P_X^1, P_Y^1) the solution of system (3.8) with $\varepsilon' = 0$ and initial condition $(0,0,0,0)$ and denote by $(\mathcal{X}^1, \mathcal{Y}^1, \mathcal{P}_X{}^1, \mathcal{P}_Y{}^1)$ the solution of (3.8) with $\varepsilon' \neq 0$ and initial condition $(0,0,0,0)$. Finally, define four functions of s, g_1, g_2, g_3, g_4, by

$$\mathcal{X}^1 = X^1 + \varepsilon' g_1, \quad \mathcal{P}_X{}^1 = P_X^1 + \varepsilon' g_3,$$
$$\mathcal{Y}^1 = Y^1 + \varepsilon' g_2, \quad \mathcal{P}_Y{}^1 = P_Y^1 + \varepsilon' g_4.$$

Combining the equations satisfied by X^1, Y^1, P_X^1, P_Y^1 and $\mathcal{X}^1, \mathcal{Y}^1, \mathcal{P}_X{}^1, \mathcal{P}_Y{}^1$, we find

that (g_1, g_2, g_3, g_4) satisfy the following system

$$\begin{cases} \frac{dg_1}{ds} &= g_3 \\ \frac{dg_2}{ds} &= 2\gamma^2 (X^0)^3 ((X^0)^2 + 2g_1) \\ \frac{dg_3}{ds} &= -\gamma^2 (X^0)^2 (6g_1 + 5(X^0)^2) \\ \frac{dg_4}{ds} &\equiv 0, \end{cases} \tag{3.12}$$

with initial conditions $g_1(0) = g_2(0) = g_3(0) = 0$. Notice that the solution of $((3.7),(3.8),(3.12))$ with initial condition $(0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0)$ is

$$(-X^0, Y^0, -P_X^0, P_Y^0, -X^1, Y^1, -P_X^1, P_Y^1, g_1, -g_2, g_3).$$

Moreover, one can compute numerically that $g_1(\sqrt{2}\mathcal{K}/\sqrt{\gamma}) \sim -\pi/(2\gamma)$, $g_2(\sqrt{2}\mathcal{K}/\sqrt{\gamma}) \sim -\pi/(4\gamma)$.

Recall that from Proposition 3.6, the cut time on the geodesic starting at the origin for the nilpotent approximation with initial covector $(1, 1)$ is $\sqrt{2}\mathcal{K}/\sqrt{\gamma}$, which corresponds to the first intersection with the symmetric geodesic whose initial covector is $(-1, 1)$. Let us compute the exponential map at the origin for the structure defined in (3.6) at time $t = \sqrt{2}\mathcal{K}\eta_0/\sqrt{\gamma}$ close to the initial condition η_0, that is, for $\eta = \eta_0 + c\eta_0^2 + o(\eta_0^2)$. Making Taylor expansions in terms of η_0, the geodesic with initial covector $p_x(0) = 1, p_y(0) = 1/\eta^2$ is

$$\begin{aligned} x &= \eta X^0(\tfrac{\sqrt{2}\mathcal{K}\eta_0}{\sqrt{\gamma}\eta}) + \eta_0^2(X^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \varepsilon' g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})) + o(\eta_0^2), \\ y &= \eta^3 Y^0(\tfrac{\sqrt{2}\mathcal{K}\eta_0}{\sqrt{\gamma}\eta}) + \eta_0^4(Y^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \varepsilon' g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})) + o(\eta_0^4). \end{aligned}$$

Since $X^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) = \dot{Y}^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) = 0$ and $\dot{X}^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) = -1$, we deduce

$$\begin{aligned} x &= \eta_0^2(X^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \varepsilon' g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \tfrac{\sqrt{2}\mathcal{K}c}{\sqrt{\gamma}}) + o(\eta_0^2), \\ y &= \eta_0^3 Y^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \eta_0^4(Y^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \varepsilon' g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + 3cY^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})) + o(\eta_0^4). \end{aligned}$$

On the other hand, the geodesic with initial covector $p_x(0) = -1, p_y = 1/\bar{\eta}^2$ where $\bar{\eta} = \eta_0 + c'\eta_0^2 + o(\eta_0^2)$ is

$$\begin{aligned} x &= \eta_0^2(-X^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \varepsilon' g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) - \tfrac{\sqrt{2}\mathcal{K}c'}{\sqrt{\gamma}}) + o(\eta_0^2), \\ y &= \eta_0^3 Y^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + \eta_0^4(Y^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) - \varepsilon' g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + 3c'Y^0(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})) + o(\eta_0^4). \end{aligned}$$

These expressions are affine with respect to parameters c and c', up to order 2 for x and 4 for y in the variable η_0. The two geodesics with initial covectors $p_x(0) = 1, p_y(0) = 1/\eta^2$ and $p_x(0) = -1, p_y(0) = 1/\bar{\eta}^2$ intersect for

$$\begin{aligned} c + c' &= -\frac{\sqrt{2\gamma} X^1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})}{\mathcal{K}} + o(1), \\ c' - c &= \frac{\sqrt{2\gamma}\varepsilon' g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})}{\mathcal{K}} + o(1) \end{aligned}$$

where $o(1)$ denotes any function going to 0 with η_0. Solving the system for c, c', the intersection occurs for

$$c = -\frac{\sqrt{\gamma}}{\sqrt{2}\mathcal{K}}\left(\varepsilon' g_2\left(\frac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}\right) + X^1\left(\frac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}\right)\right) + o(1)$$

$$c' = \frac{\sqrt{\gamma}}{\sqrt{2}\mathcal{K}}\left(\varepsilon' g_2\left(\frac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}\right) - X^1\left(\frac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}\right)\right) + o(1),$$

which implies that the cut point is

$$x_{\text{cut}} = \eta_0^2 \varepsilon'(g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) - g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}})) + o(\eta_0^2),$$

$$y_{\text{cut}} = \eta_0^3 \tfrac{\sqrt{2}\mathcal{K}}{3\sqrt{\gamma}} + o(\eta_0^3).$$

Since $g_1(\sqrt{2}\mathcal{K}/\sqrt{\gamma}) - g_2(\sqrt{2}\mathcal{K}/\sqrt{\gamma}) \sim \pi/(4\sqrt{\gamma})$, if $\varepsilon' \neq 0$, the upper branch of the cut locus at $(0,0)$ accumulates as the set $\{(x,y) \mid y > 0, y^2 = \alpha_1 x^3\}$, where

$$\alpha_1 = \frac{2\mathcal{K}^2}{9\gamma\varepsilon'^3(g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) - g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}))^3} \sim -\frac{128\mathcal{K}^2\gamma^2}{9\varepsilon'^3\pi^3}.$$

Similar computations show that the lower branch of the cut locus from $(0,0)$ accumulates as the set $\{(x,y) \mid y < 0, y^2 = \alpha_2 x^3\}$, where

$$\alpha_2 = \frac{2\mathcal{K}^2}{9\gamma\varepsilon'^3(g_1(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}) + g_2(\tfrac{\sqrt{2}\mathcal{K}}{\sqrt{\gamma}}))^3} \sim -\frac{128\mathcal{K}^2\gamma^2}{243\varepsilon'^3\pi^3}.$$

$\blacksquare$

3.3.4   The cut locus from the singular set

In this section we prove Proposition 3.3 starting from the local representation (3.4). Since $\psi(0) > 0$, the singular set $\mathcal{Z}$ is locally contained in the upper half plane $\{(x,y) \mid y \geq 0\}$.

Locally, the singular set separates M into two domains $\{(x,y) \mid y - x^2\psi(x) > 0\}$ and $\{(x,y) \mid y - x^2\psi(x) < 0\}$. Notice that since we are computing $\mathbf{K}_{\mathcal{Z}}$, the cut locus from $\mathcal{Z}$, we have $\mathbf{K}_{\mathcal{Z}} \cap \mathcal{Z} = \emptyset$. Moreover the only point of $\mathcal{Z}$ where $\mathbf{K}_{\mathcal{Z}}$ can accumulate is the tangency point, the distribution being transversal to $\mathcal{Z}$ at all other points of $\mathcal{Z}$. Hence, locally, $\mathbf{K}_{\mathcal{Z}}$ is the union of two parts, $\mathbf{K}_{\mathcal{Z}}^+$ lying in the upper domain and $\mathbf{K}_{\mathcal{Z}}^-$ in the lower one.

As we shall see, the two components of $\mathbf{K}_{\mathcal{Z}}$ have different natures: in the upper domain the geodesic starting at a point $(a, a^2\psi(a))$ and minimizing the distance from $\mathcal{Z}$ reaches its cut point at a time of order 1 in $|a|$, when in the lower domain the geodesic starting at the same point reaches its cut point at a time of order 1 in $\sqrt{|a|}$.

Applying the PMP, geodesics for the ARS are projections on $\mathbb{R}^2$ of solutions of the Hamiltonian system associated with the function

$$H = \frac{1}{2}(p_x^2 + p_y^2(y - x^2\psi(x))^2 e^{2\xi(x,y)}),$$

that is, solutions of the system

$$
\begin{cases}
\dot{x} &= p_x \\
\dot{y} &= p_y((y - x^2\psi(x))e^{\xi(x,y)})^2 \\
\dot{p}_x &= p_y^2(y - x^2\psi(x))(2x\psi(x) + x^2\psi'(x) - (y - x^2\psi(x))\frac{\partial\xi}{\partial x}(x,y))e^{2\xi(x,y)} \\
\dot{p}_y &= -p_y^2(y - x^2\psi(x))(1 + (y - x^2\psi(x))\frac{\partial\xi}{\partial y}(x,y))e^{2\xi(x,y)}.
\end{cases}
\tag{3.13}
$$

In addition, a solution with $x(0) = a, y(0) = a^2\psi(a)$, $a \neq 0$ and minimizing the distance from $\mathcal{Z}$ must satisfy the transversality condition

$$
p_x(0) = \pm 1, \quad p_y(0) = \mp\frac{1}{2a\psi(a) + a^2\psi'(a)}.
$$

Introducing the new time variable $s = \frac{t}{\eta}$ where $\eta > 0$ is a parameter, system (3.13) becomes

$$
\begin{cases}
\frac{dx}{ds} &= \eta p_x \\
\frac{dy}{ds} &= \eta p_y((y - x^2\psi(x))e^{\xi(x,y)})^2 \\
\frac{dp_x}{ds} &= \eta p_y^2(y - x^2\psi(x))(2x\psi(x) + x^2\psi'(x) - (y - x^2\psi(x))\frac{\partial\xi}{\partial x}(x,y))e^{2\xi(x,y)} \\
\frac{dp_y}{ds} &= -\eta p_y^2(y - x^2\psi(x))(1 + (y - x^2\psi(x))\frac{\partial\xi}{\partial y}(x,y))e^{2\xi(x,y)}.
\end{cases}
\tag{3.14}
$$

The proof of the result splits into two steps, where we describe first $\mathbf{K}_{\mathcal{Z}}^{+}$ and then $\mathbf{K}_{\mathcal{Z}}^{-}$. In each step we proceed as follows: first we compute jets of the exponential map; second we try to identify which geodesics intersect at the same time t; finally, we check that the conjugate time of these geodesics is bigger than t.

The upper part of the cut locus

We consider the geodesic starting from a point of $\mathcal{Z}$ with initial condition

$$
x(0) = a > 0, \ y(0) = a^2\psi(a), \ p_x(0) = -1, \ p_y(0) = \frac{1}{2a\psi(a) + a^2\psi'(a)}, \tag{3.15}
$$

i.e., the geodesic realizing locally the distance from $\mathcal{Z}$ and entering the upper domain. Taking $\eta = a$, one can check that if x, y, p_x, p_y have orders $1, 2, 0, -1$ in η respectively, then the dynamics has the same or higher orders. As a consequence, since the initial conditions respect these orders, one can compute jets with respect to η of the solution of system (3.14) under the form

$$
\begin{array}{llll}
x(s) &= \eta x_0(s) + \eta^2 x_1(s) + \eta^3 \bar{x}(\eta, s) & p_x(s) &= p_{x0}(s) + \eta p_{x1}(s) + \eta^2 \bar{p}_x(\eta, s) \\
y(s) &= \eta^2 y_0(s) + \eta^3 y_1(s) + \eta^4 \bar{y}(\eta, s) & p_y(s) &= \eta^{-1}p_{y0}(s) + p_{y1}(s) + \eta \bar{p}_y(\eta, s)
\end{array}
$$

where $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ are smooth functions. Using (3.15), the initial conditions are given by

$$
\begin{array}{llll}
x_0(0) = 1, & x_1(0) = 0, & p_{x0}(0) = -1, & p_{x1}(0) = 0 \\
y_0(0) = 1 & y_1(0) = \psi'(0), & p_{y0}(0) = \frac{1}{2}, & p_{y1} = -\frac{3}{4}\psi'(0),
\end{array}
$$

and from system (3.14) we easily get

$$
x_0(s) = 1 - s, \ x_1(s) \equiv 0, \ y_0(s) \equiv 1, \ y_1(s) \equiv \psi'(0),
$$

whence

$$x(t) = a - t + a^3 \bar{x}(a, t/a), \quad y(t) = a^2 + a^3 \psi'(0) + a^4 \bar{y}(a, t/a).$$

Similarly, the solution of (3.13) with initial condition

$$x(0) = a < 0, \ y(0) = a^2 \psi(a), \ p_x(0) = 1, \ p_y(0) = \frac{1}{2a\psi(a) + a^2\psi'(a)}, \qquad (3.16)$$

satisfies

$$x(t) = a + t + a^3 \bar{x}(a, t/a), \quad y(t) = a^2 + a^3 \psi'(0) + a^4 \bar{y}(a, t/a).$$

This allows to prove that, for any $c > 0$, at t fixed, $\frac{\partial x(t)}{\partial a} > \frac{1}{2}$ for $0 < |\frac{t}{a}| < c$ and a small enough. Hence two geodesics starting with initial conditions a and $\bar{a}$ of the same sign do not intersect at time t if a and $\bar{a}$ are small enough and $|\frac{t}{a}|$ and $|\frac{t}{\bar{a}}|$ are smaller than c. As concerns the conjugate locus, the Jacobian of the map $(a, t) \mapsto (x(t), y(t))$ (up to a multiplicative constant) is equal to $2a + 3a^2\psi'(0) + a^3\Xi(a, \frac{t}{a})$ where Ξ is a smooth function. This implies that for $|\frac{t}{a}| < c$ and a small enough, the Jacobian is nonzero, whence t is not a conjugate time.

Let us prove that a geodesic with initial condition $a > 0$ small enough intersects exactly one geodesic with an initial condition $\bar{a} < 0$ of the same length at $t \sim a$. Then, the upper part of the cut locus is the set given by the intersection of these two geodesics, as the parameter a varies in a small right neighborhood of zero. If two geodesics intersect at the same time, one with $a > 0$ and the other with $\bar{a} < 0$, then

$$a^2 + a^3\psi'(0) + o(a^3) = \bar{a}^2 + \bar{a}^3\psi'(0) + o(\bar{a}^3).$$

Thus we get that $\bar{a} = -a - a^2\psi'(0) + o(a^2)$, the cut time is $t_{cut} = a + \frac{1}{2}a^2\psi'(0) + o(a^2)$ and the cut point is

$$x_{cut} = -\frac{\psi'(0)}{2}a^2 + o(a^2), \quad y_{cut} = a^2 + o(a^2). \qquad (3.17)$$

It is easy to see that when the two geodesics intersect, the corresponding fronts are transverse to each other whence the upper branch of the cut locus from $\mathcal{Z}$ is a smooth curve, in a sufficiently small neighborhood of $(0, 0)$. From (3.17) we deduce moreover that the tangent vector to $\mathbf{K}_{\mathcal{Z}}^+$ at $(0, 0)$ is $(-\psi'(0)/2, 1)$, which does not belong to the distribution at $(0, 0)$.

The lower part of the cut locus

Reasoning as in the previous section, we consider the geodesic starting from a point of $\mathcal{Z}$ with initial condition

$$x(0) = a > 0, \ y(0) = a^2\psi(a), \ p_x(0) = 1, \ p_y(0) = -\frac{1}{2a\psi(a) + a^2\psi'(a)}, \qquad (3.18)$$

i.e., the geodesic realizing locally the distance from $\mathcal{Z}$ and entering the lower domain. Taking $\eta = \sqrt{a}$, one can check that if x, y, p_x, p_y have orders in η higher or equal

to $1, 3, 0, -2$, respectively, then the dynamics has the same or higher orders. As a consequence, since the initial conditions respect these orders, one can compute jets with respect to η of the solution of system (3.14) under the form

$$
\begin{array}{llll}
x(s) & = & \eta x_0(s) + \eta^2 x_1(s) + \eta^3 \bar{x}(\eta, s) & \quad p_x(s) & = & p_{x0}(s) + \eta p_{x1}(s) + \eta^2 \bar{p}_x(\eta, s) \\
y(s) & = & \eta^3 y_0(s) + \eta^4 y_1(s) + \eta^5 \bar{y}(\eta, s) & \quad p_y(s) & = & \eta^{-2} p_{y0}(s) + \eta^{-1} p_{y1}(s) + \bar{p}_y(\eta, s),
\end{array}
$$

where $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ are smooth functions. From the initial conditions (3.18), we deduce

$$
\begin{array}{llll}
x_0(0) = 0, & x_1(0) = 1, & p_{x0}(0) = 1, & p_{x1}(0) = 0, \\
y_0(0) = 0, & y_1(0) = 1, & p_{y0}(0) = -\frac{1}{2}, & p_{y1}(0) = 0,
\end{array}
$$

and by (3.14) we get that $x_0, x_1, y_0, y_1, p_{x0}, p_{x1}, p_{y0}, p_{y1}$ satisfy systems (3.7) (3.8). Thus $p_{y0} \equiv -\frac{1}{2}$ and by Proposition 3.6 it follows that

$$
x_0(s) = -\frac{\sqrt{2}}{\sqrt{\gamma}} \mathrm{cn}\,(\mathcal{K} + \sqrt{\gamma}s),
$$

$$
y_0(s) = -\frac{2}{3\sqrt{\gamma}}(\sqrt{\gamma}s + 2\mathrm{sn}\,(\mathcal{K} + \sqrt{\gamma}s)\mathrm{cn}\,(\mathcal{K} + \sqrt{\gamma}s)\mathrm{dn}\,(\mathcal{K} + \sqrt{\gamma}s)).
$$

Recall that the Jacobi functions cn, sn are $4\mathcal{K}$-periodic, when dn is $2\mathcal{K}$-periodic.

Denote by $x_{10}, y_{10}, p_{x10}, p_{y10}$ the solution of system (3.8) with $\varepsilon' = 0$. Define g_1, g_2, g_3, g_4 by

$$
\begin{array}{llll}
x_1 & = & x_{10} + \varepsilon' g_1, & p_{x1} & = & p_{x10} + \varepsilon' g_3, \\
y_1 & = & y_{10} + \varepsilon' g_2, & p_{y1} & = & p_{y10} + \varepsilon' g_4.
\end{array}
$$

It is easy to see that the g_i satisfy

$$
\begin{cases}
\dot{g}_1 & = & g_3 \\
\dot{g}_2 & = & -\gamma^2 x_0^3 (2g_1 + x_0^2) \\
\dot{g}_3 & = & -\frac{1}{4}\gamma^2 x_0^2 (6g_1 + 5x_0^2) \\
g_4 & \equiv & 0,
\end{cases}
\tag{3.19}
$$

and the initial conditions are $g_1(0) = g_2(0) = g_3(0) = 0$. Notice moreover that, if

$$
(x_0, y_0, p_{x0}, p_{y0}, x_{10}, y_{10}, p_{x10}, p_{y10}, g_1, g_2, g_3)
$$

is the solution of (3.7), (3.8), (3.19) with initial condition $(0, 0, 1, -1/2, 1, 1, 0, 0, 0, 0, 0)$ then the solution of (3.7), (3.8), (3.19) with initial condition $(0, 0, -1, -1/2, -1, 1, 0, 0, 0, 0, 0)$ is

$$
(-x_0, y_0, -p_{x0}, p_{y0}, -x_{10}, y_{10}, -p_{x10}, p_{y10}, g_1, -g_2, g_3),
$$

which is a geodesic starting from a point of $\mathcal{Z}$ with $a < 0$.

We are going to prove that there exists $\delta > 0$ such that if $\eta \neq 0$ is small enough and $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$ then, at t fixed, $\frac{\partial x(t)}{\partial a} > 0$. This implies in particular that two geodesics with initial conditions a and $\bar{a}$ of the same sign do not intersect at t if a and $\bar{a}$ are small enough and $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$. Indeed we have

$$
\frac{\partial x(t)}{\partial \eta} = x_0(t/\eta) - \frac{t}{\eta}\dot{x}_0(t/\eta) + \eta\left(2x_1(t/\eta) - \frac{t}{\eta}\dot{x}_1(t/\eta)\right) + \eta^2 x_r(\eta, t/\eta)
$$

where x_r is a smooth function. The function $f : u \mapsto x_0(u) - u\dot{x}_0(u)$ satisfies $f(0) = 0$ and

$$f'(u) = -u\ddot{x}_0(u) = \frac{1}{2}u\gamma^2 x_0^3(u) > 0 \text{ for } u \in (0, 2\mathcal{K}/\sqrt{\gamma}).$$

Hence $f(u) > 0$ for every $0 < u \le 2\mathcal{K}/\sqrt{\gamma}$. Since $2x_1(\frac{t}{\eta}) - \frac{t}{\eta}\dot{x}_1(\frac{t}{\eta}) = 2$ for $t = 0$, there exists $\tilde{\delta} > 0$ such that if $0 < \frac{t}{\eta} < \tilde{\delta}$ and η is small enough then $\frac{\partial x(t)}{\partial \eta} > 0$. Moreover, computing the function f, it is not hard to see that given $\varepsilon > 0$ small enough there exists δ such that $f(t/\eta) > \epsilon$ for every $\delta < t/\eta < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$. Hence, for η small enough and $\delta < t/\eta < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$ we have $\frac{\partial x(t)}{\partial \eta} > 0$. Jointly with the previous remark this implies that $\frac{\partial x(t)}{\partial \eta} > 0$ for every η small enough and $0 < t/\eta < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$. Thus two geodesics corresponding to initial conditions a and $\bar{a}$ of the same sign such that η is small enough cannot intersect at the time t satisfying $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$.

As concerns the conjugate locus, the Jacobian of $(\eta, s) \mapsto (x(s), y(s))$ (up to a multiplicative constant) is equal to $\eta^3(J_0(s) + \eta J_1(s) + \eta^2 J_2(s, \eta))$, where $J_0(s) = x_0(s)\dot{y}_0(s) - 3y_0(s)\dot{x}_0(s)$, $J_1(0) = -4\text{sign}(\dot{x}_0(0))$, and J_2 is a smooth function. It was proved in [12] that J_0 does not vanish on $(0, \bar{s})$ with $\bar{s} > 2\mathcal{K}/\sqrt{\gamma}$. Moreover $J_1(0)$ has the same sign as the function J_0 on the interval $(0, \bar{s})$. Hence possibly reducing δ, the Jacobian never vanishes on the interval $[0, \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta)$. This allows to conclude that if $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$ and a small enough, t is not a conjugate time.

Let us prove that for $a > 0$ small enough there exists exactly one $\bar{a} < 0$ such that the geodesics starting with the initial conditions a and $\bar{a}$ are optimal until their intersection at a time t satisfying $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$. In order to do that, we start by finding the good candidates. For $a > 0$, the corresponding geodesic parametrized by s is

$$
\begin{aligned}
x_+(s) &= \eta x_0(s) + \eta^2(x_{10}(s) + \varepsilon' g_1(s)) + o(\eta^2), \\
y_+(s) &= \eta^3 y_0(s) + \eta^4(y_{10}(s) + \varepsilon' g_2(s)) + o(\eta^4),
\end{aligned}
$$

when for $\bar{a} < 0$ it is

$$
\begin{aligned}
x_-(s) &= -\eta x_0(s) + \eta^2(-x_{10}(s) + \varepsilon' g_1(s)) + o(\eta^2), \\
y_-(s) &= \eta^3 y_0(s) + \eta^4(y_{10}(s) - \varepsilon' g_2(s)) + o(\eta^4).
\end{aligned}
$$

Let us estimate these geodesics for $t_0 = \frac{2\mathcal{K}}{\sqrt{\gamma}}\eta_0$, $\eta_+ = \eta_0 + c_+\eta_0^2 + o(\eta_0^2)$ and $\eta_- = \eta_0 + c_-\eta_0^2 + o(\eta_0^2)$. One computes easily that

$$
\begin{aligned}
s_+ &= \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_+\eta_0 + o(\eta_0)), \\
s_- &= \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_-\eta_0 + o(\eta_0)),
\end{aligned}
$$

and

$$x_+(t_0) = \eta_0^2\left(-c_+\frac{2\mathcal{K}}{\sqrt{\gamma}} + x_{10}(2\mathcal{K}/\sqrt{\gamma}) + \varepsilon' g_1(2\mathcal{K}/\sqrt{\gamma})\right) + o(\eta_0^2),$$

$$y_+(t_0) = -\eta_0^3 \frac{4\mathcal{K}}{3\sqrt{\gamma}} + \eta_0^4 \left(-\frac{4\mathcal{K}c_+}{\sqrt{\gamma}} + y_{10}(2\mathcal{K}/\sqrt{\gamma}) + \varepsilon' g_2(2\mathcal{K}/\sqrt{\gamma}) \right) + o(\eta_0^4),$$

$$x_-(t_0) = \eta_0^2 \left(c_- \frac{2\mathcal{K}}{\sqrt{\gamma}} - x_{10}(2\mathcal{K}/\sqrt{\gamma}) + \varepsilon' g_1(2\mathcal{K}/\sqrt{\gamma}) \right) + o(\eta_0^2),$$

$$y_-(t_0) = -\eta_0^3 \frac{4\mathcal{K}}{3\sqrt{\gamma}} + \eta_0^4 \left(-\frac{4\mathcal{K}c_-}{\sqrt{\gamma}} + y_{10}(2\mathcal{K}/\sqrt{\gamma}) - \varepsilon' g_2(2\mathcal{K}/\sqrt{\gamma}) \right) + o(\eta_0^4).$$

Hence, these two geodesics intersect for

$$c_+ = \sqrt{\gamma} \, \frac{\varepsilon' g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) - 2x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{4\mathcal{K}},$$

$$c_- = -\sqrt{\gamma} \, \frac{\varepsilon' g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + 2x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{4\mathcal{K}}.$$

The corresponding point is

$$x_{int}(t_0) = \eta_0^2 \varepsilon' \, \frac{2g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{2} + o(\eta_0^2),$$

$$y_{int}(t_0) = -\eta_0^3 \frac{4\mathcal{K}}{3\sqrt{\gamma}} + o(\eta_0^3),$$

and the intersection time satisfies

$$\frac{t_0}{\eta_+} = \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_+\eta_0 + o(\eta_0)) < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta \tag{3.20}$$

for a small enough.

The inequality (3.20) proves that locally a geodesic cannot loose optimality by reaching the conjugate locus or by intersecting a geodesic with an initial condition a of the same sign.

We claim that the two geodesics are optimal until time t_0. The idea of the proof is that if one of the two geodesic looses optimality at $\bar{t} < t_0$, then there exists another geodesic optimal until $\bar{t}$ intersecting it at $\bar{t}$. But if this is the case, this new geodesic has lost optimality before by computation similar to the one giving rise to the inequality (3.20). As a consequence of these arguments, we can conclude that $(x_{int}(t_0), y_{int}(t_0))$ is a cut point.

One can compute numerically that $2g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) \neq 0$ which implies that $\mathbf{K}_{\mathcal{Z}}^-$ accumulates at $(0,0)$ as the lower branch of a cusp. Finally, since the fronts corresponding to $a > 0$ and $a < 0$ are transverse at the cut points, $\mathbf{K}_{\mathcal{Z}}^-$ is locally a curve. ∎

3.3.5 Local representation at tangency points

In this section we prove the existence of a crest of the curvature passing through a tangency point q which is smooth, has tangent direction at q transverse to the distribution and admits a canonical parametrization. Notice that, taking the completely reduced local representation associated with the curve given by Theorem 3.4,

if the parameter $\psi'(0) + \psi(0)\partial_x\xi(0,0)$ is nonzero, then the component of $\mathbf{K}_{\mathcal{Z}}$ arriving transversally to the distribution and the crest of the curvature are transversal to each other.

Proof of Theorem 3.4. Choose a coordinate system as in Section 3.3.4 for which we have a local representation of the type (F3) satisfying $\psi(0) = 1$. By construction, K is well defined outside the singular set $\mathcal{Z}$.

The crests or valleys of K are implicitly defined by the equation

$$G(\nabla(||\nabla K||^2), (\nabla K)^\perp) = 0. \tag{3.21}$$

Computing the left hand side of equation (3.21), we find that

$$G(\nabla(||\nabla K||^2), (\nabla K)^\perp) = \frac{e^{2\xi(x,y)}h(x,y)}{(y - x^2\psi(x))^8},$$

where h is a smooth function. Hence, equation (3.21) is equivalent to $h(x,y) = 0$. The development of h at the point $(0,0)$ is

$$h(x,y) = C\left(y^4(10x + y(3\psi'(0) - 2\xi_x(0,0))) + \sum_{i=0}^{6} a_i(x,y)x^i y^{6-i}\right),$$

where C is a nonzero constant and a_i are smooth functions. Let us show that there exists a smooth function $b : I \to \mathbb{R}$ defined on a neighborhood I of 0 such that after the coordinate change

$$X = 10x + y(3\psi'(0) - 2\xi_x(0,0)) - b(y)y^2, \quad Y = y,$$

we have $h(x(X,Y), y(X,Y)) = X\overline{h}(X,Y)$. In the new coordinate system, we have

$$h(x(X,Y), y(X,Y)) = C(Y^4 X + F(X,Y)),$$

where

$$F(X,Y) = \frac{Y^6}{10}b(Y) + \sum_{i=0}^{6} a_i(x(X,Y),Y)(x(X,Y))^i Y^{6-i}.$$

In order X to be factorizable in F, we require that $F(0,Y) \equiv 0$. Since $F(0,Y) = Y^6 R(b(Y),Y)$, where

$$R(b(Y),Y) = \frac{b(Y)}{10} + \sum_{i=0}^{6} \frac{a_i(x(0,Y),Y)}{10^i}(-3\psi'(0) + 2\xi_x(0,0) - b(Y)Y)^i,$$

it follows that $F(0,Y) \equiv 0$ if and only if there exists a smooth function b defined on a neighborhood of 0 such that $R(b(Y),Y) \equiv 0$. Let $\overline{b} = -10\sum_{i=0}^{6}\frac{a_i(0,0)}{10^i}(-3\psi'(0) + 2\xi_x(0,0))^i$. Then, since $R(b,Y)$ is smooth, $R(\overline{b},0) = 0$, $R_b(\overline{b},0) = 1/10$, we apply the Implicit Function Theorem to find a smooth function $b(Y)$ with the properties above. Therefore, coming back to the (x,y) coordinates we have shown that

$$h(x,y) = C(10x + y(3\psi'(0) - 2\xi_x(0,0)) + b(y)y^2)(y^4 + \tilde{F}(x,y)),$$

where $\tilde{F}$ is smooth and b is the function built above. The last equation implies that the curve $\{(x,y) \mid 10x + y(3\psi'(0) - 2\xi_x(0,0)) + b(y)y^2 = 0\}$ is a connected component of the set defined by equation (3.21), it is smooth, it passes through $(0,0)$ and its tangent line at $(0,0)$ is

$$x = \frac{1}{10}(2\xi_x(0,0) - 3\psi'(0))y,$$

that is transversal to the distribution at $(0,0)$.

We are left to find a canonical parametrization on the given curve. Notice that the limit of ∇K as (x,y) goes to $(0,0)$ does not exist, since the curvature does not converge at the tangency point. Nevertheless, it happens that if (x,y) tends to $(0,0)$ along a curve that approaches the origin with tangent direction $(2/10\xi_x(0,0) - 3/10\psi'(0), 1)$, then ∇K converges. Hence, we can choose the parametrization $s \mapsto \Gamma(s) = (x(s), y(s))$ such that $G(\nabla K, \dot{\Gamma}(s)) \equiv 1$, equivalently

$$-\partial_x K(x(s), y(s))(y(s)(3\psi'(0) - 2\xi_x(0,0)) + b(y(s))y(s)^2) + 10\partial_y K(x(s), y(s))\dot{y}(s) = 10.$$

∎

Proof of Proposition 3.5. Starting from the parametrized curve $s \mapsto \Gamma(s)$ given in the proof of Theorem 3.4 and following the procedure in the proof of Lemma 1.4 (here we do not assume $\psi(0) = 1$), we end up with a local representation (F3) where the curve $s \mapsto \Gamma(s)$ is the vertical axis. Hence, imposing $\dot{\Gamma}(0)$ to be vertical, the functions ψ, ξ satisfy

$$2\xi_x(0,0)\psi(0) - 3\psi'(0) = 0.$$

∎

On vector fields with a discontinuity of divide-by-zero type and applications

In this chapter we consider vector fields of the type

$$W(x) = \frac{1}{f(x)^r} V(x), \quad x \in \mathbb{R}^n, \ r > 0 \tag{4.1}$$

where f is a C^s-smooth function such that the set $\Gamma = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ is a regular submanifold, V is a C^s-smooth vector field, and $n \geq 2$. The field W is C^s-smooth on $\Omega \setminus \Gamma$, but at points of Γ formula (4.1) gives a discontinuity of divide-by-zero type. Due to their large number of applications (e.g., in mechanics with dry friction and control theory, see [22]), discontinuous vector fields (or, equivalently, differential equations with discontinuous right-hand sides) have been widely studied.

Although this problem seems at first sight rather theoretical and not natural, it is motivated by a large number of applications. Indeed, many variational problems in differential geometry and calculus of variations are characterized by Lagrangian (or Hamiltonian) functions that are smooth at all points except for a regular hyper-surface Γ. The vector field corresponding to the Euler–Lagrange equations of such problems is divergence-free and takes the form (4.1). The simplest example is the equation of geodesic lines on the cuspidal edge embedded in the Euclidean space or on the plane with the Klein metric, that is used in the model of the Lobachevsky plane.

In this chapter we establish some general facts about vector fields of the form (4.1) under two special assumptions that allow to infer some properties of the phase portraits of the vector fields V and W. Our results lead to find normal forms for the direction fields corresponding to geodesic flow metric structures with singularities. Three applications are given.

The chapter is orginized as follows. In Section 4.1 we prove several simple theorems about vector (and direction) fields of the form (4.1) under some special assumptions without any additional hypothesis on V. In particular, these results show the key role of singular points of the field V in the applications. In Section 4.2 we give a brief survey of the theory of normal forms at non-isolated singular points of smooth vector fields. We restrict to the case where the components of the vector field belong to the ideal generated by two of them in the ring of smooth functions. As far as we know, the first work devoted to the analysis of local normal forms for such fields is due to F. Takens [44]. Later, the problem was deeply investigated in finite smooth [36], C^∞-smooth [41], and analytic categories [45, 46].[1] In this survey,

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[1]Remark that the finite smooth classification is based on the general results by V.S. Samovol

we deal only with finite and C^∞-smooth classifications, which are simpler than the analytic one. In the last section we apply the results to the problem of geodesic flow generated by three different types of singular metrics on surfaces. Firstly, we consider pseudo-Riemannian metrics, i.e., metrics that degenerate (change their signature) on a curve, see also [38]. Secondly, we analyse metrics of Klein type, that are positive definite but have a singularity of divide-by-zero type, see [39]. Finally, we consider almost-Riemannian structures in a neighborhood of Grushin points. Two other examples can be found in [37, 40].

Unless specified, the results given in this chapter are to be found in [25].

4.1 Basic Theorems

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with coordinates $x = (x_1, \ldots, x_n)$. Let $f \in C^s(\Omega, \mathbb{R})$, $s \geq 2$, be such that the equation $f(x) = 0$ defines a regular hypersurface $\Gamma \subset \Omega$, i.e., at all points $x \in \Gamma$ the condition $\nabla f(x) \neq 0$ holds, where ∇f is the gradient of f. We consider vector fields of the type

$$W(x) = f^{-r}(x)\, V(x), \tag{4.2}$$

where $V \in C^s(\Omega, \mathbb{R}^n)$ is a vector field and r is a positive real number.

The divergence of W, denoted by D_W, is infinite or undetermined on the hypersurface Γ, but it is a C^{s-1}-smooth function on $\Omega \setminus \Gamma$. Assume that the field W satisfies the following main conditions

$\lim_{x \to x_*} f^{r+1}(x) D_W(x) = 0, \quad \forall\, x_* \in \Gamma,$

$\lim_{x \to x_*} f^r(x) D_W(x) = 0, \quad \lim\limits_{x \to x_*} f^{r+1}\frac{\partial D_W}{\partial x_i}(x) = 0, \quad \forall\, x_* \in \Gamma : V(x_*) = 0, \;\; \forall i.$

For simplicity we write these assumptions in the form

$$f^{r+1} D_W|_\Gamma = 0, \tag{4.3}$$

$$f^r D_W(x_*) = 0, \quad f^{r+1}\frac{\partial D_W}{\partial x_i}(x_*) = 0, \quad \forall\, x_* \in \Gamma : V(x_*) = 0, \;\; \forall i. \tag{4.4}$$

Conditions (4.3), (4.4) are fulfilled, for instance, if the vector field W is divergence-free, i.e., $D_W \equiv 0$ for all $x \in \Omega \setminus \Gamma$.

Integral curves of the fields W and V coincide at all points $x \in \Omega \setminus \Gamma$. At the same time the field V is more suitable for analysis, since it is smooth on the whole domain Ω while the field W is discontinuous on the hypersurface $\Gamma \subset \Omega$. Our concern is to pass from the initial vector field W to the vector field V.

Theorem 4.1. *Condition (4.3) holds true if and only if Γ is an invariant hypersurface of V. The function f is a first integral of V if and only if*

$$f^r D_W(x) \equiv D_V(x). \tag{4.5}$$

Assume f to be a first integral of V and let condition (4.4) holds true. Then $D_V(x_*) = 0$ for every $x_* \in \Gamma$ such that $V(x_*) = 0$.

---

[43], and the analytic classification is based on the general results by A.D. Bryuno [19, 20, 21] and J.C. Yoccoz [47].

Proof. Using the formula of divergence in Cartesian coordinates, for every point in $\Omega \setminus \Gamma$ we get

$$f^{r+1} D_W = f D_V + f^{r+1} L_V (f^{-r}) = f D_V - r L_V f, \tag{4.6}$$

where L_V denotes the Lie derivative along the vector field V. All terms in the right hand side of equation (4.6) are C^{s-1}-smooth on Ω. Hence, taking the limit as x tends to $x_* \in \Gamma$ it follows that $f^{r+1} D_W|_\Gamma = 0$ is equivalent to $L_V f|_\Gamma = 0$.

As concerns the second statement, for every point in $\Omega \setminus \Gamma$, we have

$$f^r D_W - D_V = -r f^{-1} L_V f. \tag{4.7}$$

If identity (4.5) holds on $\Omega \setminus \Gamma$, then (4.7) implies $L_V f|_{\Omega \setminus \Gamma} = 0$. By continuity it follows $L_V f \equiv 0$ on Ω, i.e., f is a first integral of V. Conversely, assume $L_V f \equiv 0$ on Ω. Then, by (4.7) we get $(f^r D_W - D_V)|_{\Omega \setminus \Gamma} = 0$. Thus, by continuity ($D_V$ is continuous on Ω), identity (4.5) on Ω follows. Finally, combining the first equality in condition (4.4) and identity (4.5), we get the last statement of the theorem. ∎

Corollary 4.2. *Assume that condition (4.3) holds. Let γ be an integral curve of either V or W passing through the point $x_* \in \Gamma$. If $V(x_*) \neq 0$, then in a neighborhood of x_* the curve γ lies entirely in the hypersurface Γ.*

Theorem 4.1 and Corollary 4.2 explain why singular points of V play an important role. Indeed, in many applications it is necessary to find integral curves that intersect the invariant hypersurface Γ but do not belong entirely to Γ. Hence such integral curves intersect Γ only at singular points. The next theorem establishes a relation between the eigenvalues of the linearization of the vector field V at a singular point $x_* \in \Gamma$. As we shall see, in many cases such a relation is resonance.
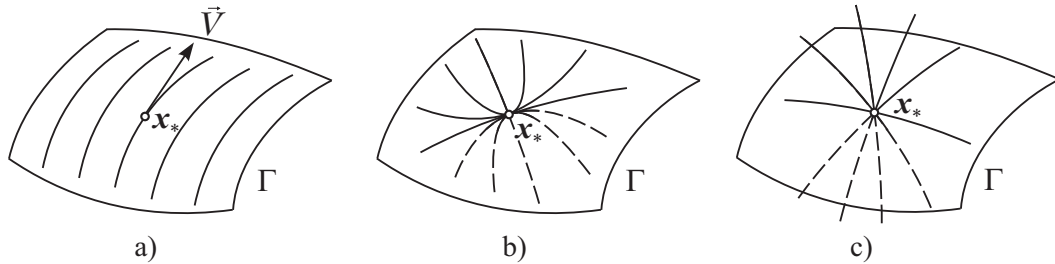
Figure 4.1: Three examples of phase portraits of the vector field V. Figure a) represents the case $V(x_*) \neq 0$ in which all integral curves belong to Γ. In Figures b), c) the case $V(x_*) = 0$ is illustrated and all integral curves except only one do not belong to Γ

Theorem 4.3. *Let $x_* \in \Gamma$ be a singular point of the field V and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the linearization of V at x_*. If conditions (4.3) and (4.4) hold true then there exists $j \in \{1, \ldots, n\}$ such that*

$$\lambda_1 + \cdots + \lambda_n = r \lambda_j. \tag{4.8}$$

There exists an eigenvector corresponding to λ_j which is transversal to Γ at x_*. The spectrum of the linearization of the restriction $V|_\Gamma$ at x_* is $\{\lambda_1, \ldots, \lambda_n\} \setminus \lambda_j$. If f is a first integral of V then $\lambda_j = 0$.

Proof. Differentiating the identity $f^{r+1}D_W = fD_V - rL_V f$ (see (4.6)) with respect to x_i we get

$$(r+1)f^r D_W \frac{\partial f}{\partial x_i} + f^{r+1}\frac{\partial D_W}{\partial x_i} = D_V \frac{\partial f}{\partial x_i} + f\frac{\partial D_V}{\partial x_i} - r\left\langle \frac{\partial V}{\partial x_i}, \nabla f \right\rangle - r\left\langle V, \nabla \frac{\partial f}{\partial x_i} \right\rangle,$$

where the triangle brackets denote the standard scalar product of vectors.

The last equality holds for all $x \in \Omega \setminus \Gamma$ and its right hand side is C^{s-2}-smooth on Ω. Taking the limit as x tends to $x_* \in \Gamma$ such that $V(x_*) = 0$ and using (4.4), we get

$$\left(D_V \frac{\partial f}{\partial x_i} - r\left\langle \frac{\partial V}{\partial x_i}, \nabla f \right\rangle \right)\bigg|_{x_*} = 0, \quad i = 1, \ldots, n.$$

Since $r \neq 0$, the last system can be written in matrix form as $Ae = \rho e$, where $A = (\frac{\partial V}{\partial x})|_{x_*}$ is the matrix of the linearization of V at the singular point x_*, the vector $e = \nabla f(x_*)$, and the number $\rho = r^{-1}D_V(x_*)$. By hypothesis, $\nabla f(x_*) \neq 0$, hence ρ is an eigenvalue of the linearization of V at x_* with corresponding eigenvector $\nabla f(x_*)$. Let $j \in \{1, \ldots n\}$ be such that $\rho = \lambda_j$. Notice that the divergence of a vector field at any singular point coincides with the trace of the linearization of this field at that point. Thus $\lambda_j = r^{-1}(\lambda_1 + \cdots + \lambda_n)$ which leads to equality (4.8). Clearly, e is transversal to Γ at x_*, whence λ_j does not belong to the spectrum of the linearization of the restriction $V|_\Gamma$.

To prove the last statement recall that if f is a first integral of V, by Theorem 4.1 $D_V(x_*) = 0$ follows. Since $D_V(x_*) = \lambda_1 + \cdots + \lambda_n$, we have equality (4.8) with $\lambda_j = 0$. ∎

Remark 4.1. Theorems 4.1 and 4.3 hold true not only for $r > 0$, but also for $r < 0$.

Let us illustrate two examples in $\mathbb{R}^3$ with coordinates (x, y, z).

Example 4.1. Consider the vector field $V(x, y, z) = (x, y, z)$. Then $D_V(x, y, z) \equiv 3$ and the unique singular point of V is the origin. The spectrum of the linearization at the origin is $(1, 1, 1)$. The field V has no non-constant first integrals, but it has the family of integral planes $ax + by + cz = 0$ passing through the origin.

Consider the vector field W given by formula (4.2) with $f(x, y, z) = ax + by + cz$. Then $D_W = (3 - r)f^{-r}$ and $f^r D_W = (3 - r)$. Condition (4.3) is satisfied, but (4.4) is fulfilled only if $r = 3$. In the case $r = 3$, we have relation (4.8) with any index $j = 1, 2, 3$.

Example 4.2. Consider the vector field $V(x, y, z) = (2x, y, 0)$. Then $D_V(x, y, z) \equiv 3$ and the set of singular points of V is the z-axis. The spectrum of the linearization at any singular point is $(2, 1, 0)$. The coordinate function z is a first integral of V and there is a family of integral surfaces given by $x - cy^2 = 0$, as c varies in $\mathbb{R}$.

Consider the vector field W given by formula (4.2) with $f(x, y, z) = z$. Then $f^r D_W = 3$ and condition (4.4) is not satisfied. This corresponds to the last claim

of Theorem 4.1. Indeed, since f is a first integral of V, condition (4.4) would imply $D_V(0,0,z) \equiv 0$.

Let now $f(x,y,z) = x - cy^2$. Defining W as in (4.2), we get $f^r D_W = (3 - 2r)$. Hence condition (4.4) holds true if and only if $r = 3/2$ and the relation (4.8) is valid with $\lambda_j = 2$.

Finally, consider the vector field W with $f(x,y,z) = y$. Then $f^r D_W = (3 - r)$, condition (4.4) holds in the case $r = 3$, and we have (4.8) with $\lambda_j = 1$.

Sometimes it is more natural to consider direction fields rather than vector fields. Recall that given a vector field V, the direction field χ associated to V is the class of vector fields φV, where $\varphi \in C^s(\Omega)$ never vanishes. Theorems 4.1, 4.3 are valid for direction fields.

Theorem 4.4. *Let $\varphi \in C^s(\Omega)$ and $\varphi(x) \neq 0$ for every $x \in \Omega$. Then Theorems 4.1, 4.3 hold true if in equation (4.2) we replace V by φV.*

Proof. It is necessary and sufficient to prove that the main assumptions (4.3) and (4.4) are invariant with respect to multiplication of the vector fields V (and consequently W) by a C^s-smooth scalar factor $\varphi \neq 0$. Indeed, $D_{\varphi W} = \varphi D_W + f^{-r} L_V \varphi$. Hence we get

$$
\begin{aligned}
f^r D_{\varphi W}\Big|_\Gamma &= (\varphi f^r D_W + L_V \varphi)\Big|_\Gamma, \\
f^{r+1} \frac{\partial D_{\varphi W}}{\partial x_i}\Big|_\Gamma &= \left(f^{r+1}\left(\frac{\partial \varphi}{\partial x_i} D_W + \varphi \frac{\partial D_W}{\partial x_i} \right) + f \frac{\partial L_V \varphi}{\partial x_i} - r \frac{\partial f}{\partial x_i} L_V \varphi \right)\Big|_\Gamma.
\end{aligned}
$$

These expressions show that conditions (4.3), (4.4) hold true for the vector fields φV, φW.

■

4.2 Fields with non-isolated singular points

We are interested in studying vector fields V of the form

$$
\dot{\xi} = v, \quad \dot{\eta} = w, \quad \dot{\zeta}_i = \alpha_i v + \beta_i w, \quad i = 1, \ldots, l, \tag{4.9}
$$

where α_i, β_i and v, w are C^∞-smooth functions of the variables $\xi, \eta, \zeta_1, \ldots, \zeta_l$. Such a kind of fields occurs in many problems, for instance, in studying implicit differential equations (see next example) and slow-fast systems.

Example 4.3. Consider the family of first-order implicit differential equations

$$
F(t, x, p) = \varepsilon, \quad p = \frac{dx}{dt}, \tag{4.10}
$$

depending on the real parameter ε not necessarily small. One effective approach (which goes back to Poincaré) consists of lifting the multi-valued direction field defined by equation (4.10) on the (t,x)-plane to a single-valued direction field χ defined by equation (4.10) in the (t, x, p)-space.

Geometrically, χ is the intersection between the contact planes $dx = pdt$ and the planes tangent to the surfaces $\{F = \varepsilon\}$ with various ε. This gives the Pfaffian system

$$F_t \, dt + F_x \, dx + F_p \, dp = 0, \quad p \, dt - dx = 0.$$

Whence the direction field χ corresponds to the vector field V given by the formula

$$\dot{t} = F_p, \quad \dot{x} = pF_p, \quad \dot{p} = -(F_t + pF_x), \qquad (4.11)$$

where a dot over a symbol means differentiation with respect to the independent variable playing the role of time. The field (4.11) has the form (4.9) with $l = 1$, where $\xi = t$, $\eta = p$, $\zeta = x$ and $v = F_p$, $w = -(F_t + pF_x)$.

In this section we recall smooth local normal forms of fields (4.9) at singular points.

The components of V belong to the ideal $I = (v, w)$ generated by two of them in the ring of germs of C^∞-smooth functions (this property is invariant with respect to the action of diffeomorphisms of the phase space). The set of singular points of V is defined by the two equations $v = w = 0$. The spectrum of the linearization of V at any singular point contains at least l zero eigenvalues, i.e., it is $(\lambda_1, \lambda_2, 0, \ldots, 0)$.

Consider the germ of V at a given singular point. Without loss of generality we may assume the singular point to be the origin of the phase space. From now on, we will always assume that $\mathrm{Re}\lambda_{1,2}(0) \neq 0$, whence the set of singular points of V is the regular center manifold of dimension l, denoted by W^c. The eigenvectors with zero eigenvalue are tangent to W^c and the eigenvectors (if they exist) corresponding to $\lambda_{1,2}(0)$ are tangent to the plane $d\zeta_i = \alpha_i \, d\xi + \beta_i \, d\eta$, $i = 1, \ldots, l$.

It is convenient to choose local coordinates $(\xi, \eta, \zeta_1, \ldots, \zeta_l)$ such that $W^c = \{\xi = \eta = 0\}$ and the linear part of V at 0 is in normal Jordan form. Then there exist C^∞-smooth functions $v_{1,2}$ and $w_{1,2}$ such that $v = \xi v_1 + \eta v_2$ and $w = \xi w_1 + \eta w_2$. The eigenvalues $\lambda_{1,2}$ at various singular points continuously depend on the variable $\zeta = (\zeta_1, \ldots, \zeta_l)$, which is a local coordinate on W^c (this dependence is C^∞-smooth at the points where $\lambda_1 \neq \lambda_2$). In the following we will always work using such coordinates.

4.2.1 Normal forms: the non-resonant case

We shall say that k functions $U^{(i)}(\xi, \eta, \zeta)$, $i = 1, \ldots, k$, are *independent by ζ* at the point 0 if their gradients with respect to the variable $\zeta = (\zeta_1, \ldots, \zeta_l)$ at 0 are linearly independent. It is not hard to see that if a function U is a first integral of V then its partial derivatives U_ξ and U_η vanish at 0. Hence the number of first integrals of V independent at 0 is not greater than l. On the other hand, the restriction of V to W^c is identically zero, hence by Shoshitaishvili's reduction theorem [8], the germ of V at 0 is orbitally topologically equivalent to

$$\dot{\xi} = \xi, \quad \dot{\eta} = \pm\eta, \quad \dot{\zeta}_i = 0, \quad i = 1, \ldots, l.$$

The trivial equations $\dot{\zeta}_i = 0$ suggest the existence of l independent first integrals of V. If a l-uple of smooth first integrals $U^{(1)}, \ldots, U^{(l)}$ independent by ζ at 0 exists,

the change of coordinates $\zeta_i \mapsto U^{(i)}$ brings the field V to the form

$$\dot{\xi} = \overline{v}, \quad \dot{\eta} = \overline{w}, \quad \dot{\zeta}_i = 0, \quad i = 1, \ldots, l,$$

where $\overline{v}$ and $\overline{w}$ are smooth functions obtained from v and w by the above change of coordinates.

The existence of l independent smooth first integrals is connected with the following concept.

Definition 4.1. The relations

$$p_1 \lambda_1 + p_2 \lambda_2 = 0, \quad p_{1,2} \in \mathbb{Z}_+, \quad p_1 + p_2 \geq 1, \tag{4.12}$$

are called resonances of first type. The minimal number $p_1 + p_2$ (i.e., p_1 and p_2 are relatively prime) is called the order of the resonance (4.12).

Consider the germ of a smooth function $U(\xi, \eta, \zeta)$ at the point 0 and its Taylor series with respect to the variables ξ, η, i.e.,

$$U(\xi, \eta, \zeta) = \sum_{p_{1,2} \in \mathbb{Z}_+} u_{p_1 p_2}(\zeta)\, \xi^{p_1} \eta^{p_2}. \tag{4.13}$$

The germ of U is called N-flat ($N \in \mathbb{N}$ or ∞) by ξ, η if $u_{p_1 p_2}(\zeta) \equiv 0$ for all $p_1 + p_2 \leq N$.

Lemma 4.5. *If between the eigenvalues $\lambda_{1,2}(0)$ there are no resonances (4.12) up to order $N \in \mathbb{N}$ inclusive, then there exist C^∞-smooth functions $U^{(1)}, \ldots, U^{(l)}$ independent by ζ at 0 such that $L_V U^{(1)}, \ldots, L_V U^{(l)}$ are N-flat by ξ, η at 0.*

Proof. We prove the lemma assuming $\lambda_{1,2}(0)$ to be real and the Jordan form of the linearization of V at 0 to be diagonal (the cases of complex eigenvalues or Jordan form with a second-order cell to be considered similarly). Then the germ of V has the form

$$\begin{aligned}
\dot{\xi} &= \xi(\lambda_1(0) + \widetilde{v}_1(\zeta) + \cdots) + \eta(\widetilde{v}_2(\zeta) + \cdots), \\
\dot{\eta} &= \xi(\widetilde{w}_1(\zeta) + \cdots) + \eta(\lambda_2(0) + \widetilde{w}_2(\zeta) + \cdots), \\
\dot{\zeta}_i &= \xi(h_1^{(i)}(\zeta) + \cdots) + \eta(h_2^{(i)}(\zeta) + \cdots), \quad i = 1, \ldots, l,
\end{aligned} \tag{4.14}$$

where all functions $\widetilde{v}_{1,2}(\zeta)$, $\widetilde{w}_{1,2}(\zeta)$, $h_{1,2}^{(i)}(\zeta)$ are C^∞-smooth and vanish at 0, and the omitted terms are C^∞-smooth containing the factor ξ or η.

The idea is to look for the functions $U^{(1)}, \ldots, U^{(l)}$ in the set of polynomials in ξ, η with coefficients smoothly depending on ζ. Namely, consider a function U in the form (4.13) with finite sum $0 \leq p_1 + p_2 \leq N$ and unknown coefficients $u_{p_1 p_2}(\zeta)$. Substituting this expression into $L_V U$ and using (4.14), we get

$$\begin{aligned}
L_V U = \sum_{p_1 + p_2 = 0}^{N} \Bigg(& u_{p_1 p_2} \xi^{p_1} \eta^{p_2} (p_1(\lambda_1(0) + \widetilde{v}_1 + \cdots) + p_2(\lambda_2(0) + \widetilde{w}_2 + \cdots)) + \\
& + u_{p_1 p_2} p_1 \xi^{p_1-1} \eta^{p_2+1}(\widetilde{v}_2 + \cdots) + u_{p_1 p_2} p_2 \xi^{p_1+1} \eta^{p_2-1}(\widetilde{w}_1 + \cdots) + \\
& + \sum_{k=1}^{l} \frac{\partial u_{p_1 p_2}}{\partial \zeta_k} \Big(\xi^{p_1+1} \eta^{p_2}(h_1^{(k)} + \cdots) + \xi^{p_1} \eta^{p_2+1}(h_2^{(k)} + \cdots) \Big) \Bigg).
\end{aligned}$$

For $L_V U$ to be 1-flat by ξ, η we set the coefficients of the monomials ξ and η equal to zero, that is,

$$\begin{cases} (\lambda_1(0) + \widetilde{v}_1)u_{10} + \widetilde{w}_1 u_{01} + \sum_{k=1}^l h_1^{(k)} \frac{\partial u_{00}}{\partial \zeta_k} & = \ 0, \\ \widetilde{v}_2 u_{10} + (\lambda_2(0) + \widetilde{w}_2)u_{01} + \sum_{k=1}^l h_2^{(k)} \frac{\partial u_{00}}{\partial \zeta_k} & = \ 0. \end{cases} \tag{4.15}$$

Since the determinant $d_1(\zeta)$ of the linear system (4.15) with respect to the unknown variables u_{10} and u_{01} is a smooth function and $d_1(0) = \lambda_1(0)\lambda_2(0) \neq 0$, the solutions u_{10} and u_{01} are smooth in a neighborhood of 0. Notice that the functions $u_{10}(\zeta)$ and $u_{01}(\zeta)$ depend on the derivatives $\frac{\partial u_{00}}{\partial \zeta_i}$, where $u_{00}(\zeta)$ is any arbitrary smooth function.

Given $n \in \{2, \ldots, N\}$ consider the coefficients of the monomials $\xi^{p_1}\eta^{p_2}$, where $p_1 + p_2 = n$. In order $L_V U$ to be n-flat, we set these coefficients to be identically zero, i.e., we solve the system

$$(p_1(\lambda_1(0) + \widetilde{v}_1) + p_2(\lambda_2(0) + \widetilde{w}_2))u_{p_1 p_2} + (p_1 + 1)\widetilde{v}_2 u_{p_1+1,p_2-1} + \\ +(p_2 + 1)\widetilde{w}_1 u_{p_1-1,p_2+1} = \varphi_{p_1 p_2}, \tag{4.16}$$

where $\varphi_{p_1 p_2}$ are polynomials of the coefficients $u_{\alpha\beta}(\zeta)$ and their first-order derivatives with $\alpha + \beta < n$ (with $u_{\alpha\beta}(\zeta) \equiv 0$ if $\alpha < 0$ or $\beta < 0$). The determinant $d_n(\zeta)$ of the linear system (4.16) with respect to variables $u_{p_1 p_2}$, $p_1 + p_2 = n$, has the form

$$d_n(\zeta) = \prod_{p_1+p_2=n} (p_1\lambda_1(0) + p_2\lambda_2(0) + \delta(\zeta)),$$

where $\delta(\zeta)$ is a smooth function vanishing at $\zeta = 0$. The absence of resonances (4.12) up to order N implies that $d_n(0) \neq 0$, whence in a neighborhood of 0 the coefficients $u_{p_1 p_2}(\zeta)$, $p_1 + p_2 = n$, smoothly depend on the functions $u_{\alpha\beta}(\zeta)$ and their first-order derivatives with $\alpha + \beta < n$.

Finally, let $u_{00}^{(1)}(\zeta), \ldots, u_{00}^{(l)}(\zeta)$ be C^∞-smooth functions independent at 0. For every index $i = 1, \ldots, l$ we define $U^{(i)}$ by solving systems (4.15) and (4.16) for $n = 2, \ldots, N$ with the initial function $u_{00} = u_{00}^{(i)}$. By construction, $U^{(1)}, \ldots, U^{(l)}$ satisfy the required conditions. ∎

Corollary 4.6. *If between the eigenvalues $\lambda_{1,2}(0)$ there are no resonances (4.12) up to order N inclusive, then the last l components $\alpha_i v + \beta_i w$ of the field (4.9) can be assumed to be N-flat by ξ, η at 0.*

Corollary 4.6 allows to get normal forms in C^k-smooth and C^∞-smooth categories. As for the C^k-smooth category, in [43] the author defines the number

$$N(k) = 2\left[(2k + 1)\frac{\max |\mathrm{Re}\lambda_{1,2}|}{\min |\mathrm{Re}\lambda_{1,2}|}\right] + 2, \quad k \in \mathbb{N},$$

the square brackets denoting the integer part of a number.

Theorem 4.7. *If between the eigenvalues $\lambda_{1,2}(0)$ there are no resonances (4.12) up to order $N(k)$, then the germ of (4.9) is C^k-smoothly equivalent to*

$$\dot{\xi} = v, \quad \dot{\eta} = w, \quad \dot{\zeta}_i = 0, \quad i = 1, \ldots, l, \tag{4.17}$$

where v, w are some new functions of ξ, η, ζ. If between the eigenvalues $\lambda_{1,2}(\zeta)$ there are no resonances (4.12) of any order for all ζ sufficiently close to 0, then the germ of (4.9) is C^∞-smoothly equivalent to (4.17).

The proof of Theorem 4.7 in the finite-smooth category is based on Lemma 4.5 and on general results from [43]. The proof in the C^∞-smooth category requires more advanced techniques (see [41] or [30]). Notice that if $\mathrm{Re}\lambda_{1,2}(0)$ have the same sign the absence of resonances (4.12) between $\lambda_{1,2}(0)$ implies the absence of resonances (4.12) between $\lambda_{1,2}(\zeta)$ for all ζ sufficiently close to 0. This is no longer true if $\mathrm{Re}\lambda_{1,2}(0)$ have different signs, except for the special case when the ratio $\lambda = \lambda_1/\lambda_2$ is constant on W^c, i.e., at all singular points.

As we shall see in the following, the normal form (4.17) can be further simplified.

Definition 4.2. The relations

$$p_1\lambda_1 + p_2\lambda_2 = \lambda_j, \quad p_{1,2} \in \mathbb{Z}_+, \quad p_1 + p_2 \geq 2, \quad j \in \{1, 2\}, \qquad (4.18)$$

are called resonances of second type. The number $p_1 + p_2$ is the *order* of resonance.

Clearly, a resonance (4.12) of order n implies a resonance (4.18) of order $n + 1$. In this section we assume the absence of resonances (4.12) up to order $N \in \mathbb{N}$ or ∞. Hence a resonance (4.18) of order $\leq N$ holds if and only if the ratio $\lambda(0) = \lambda_1(0)/\lambda_2(0)$ or its inverse belongs to $\{2, \ldots, N\}$. Combining the results from [30], [36], [41], [43], one gets the following theorems.

Theorem 4.8. *Let $k \in \mathbb{N}$ and assume that between $\lambda_{1,2}(0)$ there are no resonances (4.18) of order $N(k)$ inclusive. Then the germ of V at 0 is C^k-smoothly equivalent to*

$$\dot\xi = \alpha_1(\zeta)\xi + \alpha_2(\zeta)\eta, \quad \dot\eta = \alpha_3(\zeta)\xi + \alpha_4(\zeta)\eta, \quad \dot\zeta_i = 0, \quad i = 1, \ldots, l. \qquad (4.19)$$

Moreover, if $\lambda_{1,2}(0)$ are real and $\lambda(0) \neq 1$, the germ of V at 0 is C^k-smoothly orbitally equivalent to

$$\dot\xi = \lambda(\zeta)\xi, \quad \dot\eta = \eta, \quad \dot\zeta_i = 0, \quad i = 1, \ldots, l. \qquad (4.20)$$

Both statements hold true with $k = \infty$ if between $\lambda_{1,2}(\zeta)$ there are no resonances (4.18) of any order for all ζ sufficiently close to 0.

Theorem 4.9. *Assume that $\lambda(0) = n$ is natural. Then the germ of V at 0 is C^∞-smoothly orbitally equivalent to*

$$\dot\xi = \lambda(\zeta)\xi + \varphi(\zeta)\eta^n, \quad \dot\eta = \eta, \quad \dot\zeta_i = 0, \quad i = 1, \ldots, l. \qquad (4.21)$$

If $\varphi(0) \neq 0$, then $\varphi(\zeta)$ simplifies to 1; if $\varphi(\zeta)$ has a zero of finite order s at the origin then $\varphi(\zeta)$ simplifies to ζ^s.

The normal forms (4.19)–(4.21) show that in a small neighborhood of 0 the phase portrait of V is rather simple and V has a smooth 2-dimensional invariant foliation given by the equation $\zeta = c$ in normal coordinates. The restriction of V to each leaf $\zeta = c$ is a planar vector field with non-degenerate singular point: node, saddle, or focus.

4.2.2 Normal forms: the resonant case

Consider the case where between $\lambda_{1,2}(0)$ there is a resonance (4.12), i.e., there exist $n, m \in \mathbb{N}$ relatively prime such that

$$m\lambda_1 + n\lambda_2 = 0. \tag{4.22}$$

In this case, the proof of Lemma 4.5 for $N \geq n + m$ fails, since the determinant $d_{n+m}(\zeta)$ of the linear system (4.16) with $p_1 + p_2 = n + m$ vanishes at $\zeta = 0$, and vector field (4.9) with resonance (4.22) at 0 may not have a l-uple of smooth first integrals independent at 0.

A simple illustration (with $l = 1$ and $n = m = 1$) comes from Example 4.3. Indeed, let 0 be a singular point of the vector field V given by (4.11). Clearly, F is a first integral of V, and the derivatives F_p and F_t vanish at 0. Assume that $\lambda_1(0) + \lambda_2(0) = 0$. Since $\lambda_1 + \lambda_2 = D_V = -F_x$, we have $F_x(0) = 0$, i.e., F is not regular at 0. Let $\widehat{F}$ be another first integral of V. Then the integral curves of V are 1-graphs of solutions of the implicit equation $\widehat{F}(x, y, p) = \varepsilon$ with various ε. Hence, the previous argument with $\widehat{F}$ replacing F leads to the same conclusion. Thus the germ of V at 0 admits no regular first integrals.

The resonance (4.22) generates two infinite sequences of resonances (4.18), namely,

$$(mj + 1)\lambda_1 + nj\lambda_2 = \lambda_1, \quad mj\lambda_1 + (nj + 1)\lambda_2 = \lambda_2, \quad j \in \mathbb{N}.$$

This suggests that the formal normal form contains infinite number of terms ρ^j, $\xi\rho^j$, $\eta\rho^j$, where $\rho = \xi^m\eta^n$ is called *resonance monomial* corresponding to (4.22). The central step in the derivation of normal forms in the resonant case is the following.

Lemma 4.10. *For any $k \in \mathbb{N}$, the germ of V at 0 is C^k-smoothly equivalent to*

$$\dot{\xi} = \xi(\lambda_1(0) + \Phi_1(\rho, \zeta)), \quad \dot{\eta} = \eta(\lambda_2(0) + \Phi_2(\rho, \zeta)), \quad \dot{\zeta}_i = \rho\Psi_i(\rho, \zeta), \quad i = 1, \dots, l, \tag{4.23}$$

where $\Phi_{1,2}(\rho, \zeta)$ and $\Psi_i(\rho, \zeta)$ are polynomials in $\rho = \xi^m\eta^n$ of degrees $N(k)$ and $N(k) - 1$, respectively, with coefficients smoothly depending on ζ.

Assume that $\Psi_1(0,0) \neq 0$. Then for any $\omega_1, \dots, \omega_l \in \mathbb{R}$ the germ (4.23) has a smooth first integral $U(\rho, \zeta)$ such that

$$\Phi(\rho, \zeta)U_\rho + \Psi_1(\rho, \zeta)U_{\zeta_1} + \cdots + \Psi_l(\rho, \zeta)U_{\zeta_l} = 0 \tag{4.24}$$

$$U_\rho(0,0) = \omega_1, \quad U_{\zeta_2}(0,0) = \omega_2, \dots, U_{\zeta_l}(0,0) = \omega_l, \tag{4.25}$$

where $\Phi(\rho, \zeta) = m\Phi_1(\rho, \zeta) + n\Phi_2(\rho, \zeta)$.

The proof of Lemma 4.10 is based on the general results in [43] and can be found in [36].

From now on, we will always assume $\Psi_1(0,0) \neq 0$. This hypothesis implies the existence of $l - 1$ independent first integrals $U^{(2)}, \dots, U^{(l)}$ given by solutions of (4.24) with initial conditions (4.25) corresponding to linearly independent $(l-1)$-uples $(\omega_2, \dots, \omega_l)$. Applying the change of coordinates $\zeta_i \mapsto U^{(i)}$, $i = 2, \dots, l$, the vector field (4.23) is C^∞-smoothly equivalent to

$$\dot{\xi} = \xi(\lambda_1(0) + \Phi_1(\rho, \zeta)), \quad \dot{\eta} = \eta(\lambda_2(0) + \Phi_2(\rho, \zeta)), \quad \dot{\zeta}_1 = \rho\Psi_1(\rho, \zeta), \quad \dot{\zeta}_i = 0, \quad i = 2, \dots, l, \tag{4.26}$$

where $\Phi_{1,2}(\rho, \zeta)$ and $\Psi_1(\rho, \zeta)$ are smooth functions of ρ and ζ (not necessarily polynomials in ρ like in (4.23)), $\Phi_{1,2}(0,0) = 0$, and $\Psi_1(0,0) \neq 0$.

The first integral $U(\rho, \zeta)$ given by the solution of (4.24) with initial conditions $\omega_1 = 1, \omega_2 = \ldots = \omega_l = 0$ allows to simplify the form (4.26). Considering the restriction $\Phi(\rho, \zeta)|_{W^c} = \Phi(0, \zeta)$, we analyse two cases: $\Phi_{\zeta_1}(0,0) \neq 0$, which is generic, or $\Phi(0, \zeta) \equiv 0$, which occurs in the analysis of some concrete problems (for instance, when $n = m = 1$, this condition corresponds to divergence-free fields).

In the first case, there exists a C^∞-smooth change of coordinates that preserves the form (4.26) and brings the first integral satisfying (4.24) with initial conditions $\omega_1 = 1, \omega_2 = \ldots = \omega_l = 0$ to the form $U(\rho, \zeta) = \rho + \zeta^2$. Even if the form (4.26) cannot be further simplified, the phase portrait of V can be described using the invariant foliation $\rho + \zeta^2 = c$, see [36].

Similarly, in the second case there exists a C^∞-smooth change of coordinates that preserves the form (4.26) and brings the first integral satisfying (4.24) with initial conditions $\omega_1 = 1, \omega_2 = \ldots = \omega_l = 0$ to the form $U(\rho, \zeta) = \rho$. Using this fact, the normal form (4.26) simplifies as follows.

Theorem 4.11. *If conditions $\Psi_1(0,0) \neq 0$ and $\Phi(0, \zeta) \equiv 0$ in (4.23) hold, then the germ of V at 0 is C^∞-smoothly orbitally equivalent to*

$$\dot{\xi} = n\xi, \quad \dot{\eta} = -m\eta, \quad \dot{\zeta}_1 = \rho, \quad \dot{\zeta}_i = 0, \quad i = 2, \ldots, l. \tag{4.27}$$

The normal form (4.27) with any $n, m \in \mathbb{N}$ was established in the C^k-smooth category for arbitrary $k \in \mathbb{N}$ in [36]. It was previously proved by R. Roussarie for the partial case $n = m = 1$ in C^∞-smooth category [41]. The techniques developed in [41] can be applied to establish the normal form (4.27) with any $n, m \in \mathbb{N}$ in the C^∞-smooth category. However, to our knowledge, this result is not published.

Remark 4.2. Theorem 4.11 is not valid in the analytic case: the analytic normal form is obtained from the smooth normal form (4.27) by adding some module, see [45, 46].

Remark 4.3. The condition $\Psi_1(0,0) \neq 0$ in Lemma 4.10 can be replaced by $\Psi_i(0,0) \neq 0$ for some $i \in \{1, \ldots, l\}$. This condition holds true for germs (4.9) with resonance (4.22) having generic $(n + m)$-jet. Moreover, in order to check this condition it is sufficient to bring only the $(n + m)$-jet of (4.9) to the form (4.23).

The following example shows that the condition $\Psi_i(0,0) \neq 0$ is essential.

Example 4.4. Consider the vector fields

$$\dot{\xi} = \xi, \quad \dot{\eta} = -\eta, \quad \dot{\zeta} = 0, \tag{4.28}$$

$$\dot{\xi} = \xi, \quad \dot{\eta} = -\eta(1 + \xi\eta), \quad \dot{\zeta} = \xi\eta\zeta, \tag{4.29}$$

both having at each singular point the resonance (4.22) with $n = m = 1$, whence $\Phi(0, \zeta) \equiv 0$. Clearly, the plane $\{\zeta = 0\}$ is invariant for both the vector fields and it is transversal to the center manifold $W^c = \{\xi = \eta = 0\}$ at the origin. If the germ of either (4.28) or (4.29) were C^k-smoothly ($k \geq 2$) orbitally equivalent to the normal form (4.27), then (4.27) had a C^k-smooth invariant surface transversal to the ζ-axis,

i.e., of the form $\zeta = f(\xi, \eta)$. On the other hand, substituting the Taylor expression (of the second degree) of the function $f(\xi, \eta)$ into equation $\xi f_\xi - \eta f_\eta - \xi\eta = 0$, it is not hard to see that (4.27) can not have an invariant surface of the form $\zeta = f(\xi, \eta)$.

Remark 4.4. If $n+m$ is rather large and the ratio n/m is sufficiently close to 1, the inequality $n+m > 2[(2k+1)\max\{n/m, m/n\}]+2$ has solutions $k \in \mathbb{N}$. According to Theorem 4.8, for any such k the germ of (4.27) is C^k-smoothly orbitally equivalent to (4.20) with $\lambda(\zeta) \equiv -n/m$ or, equivalently, to the field

$$\dot{\xi} = n\xi, \quad \dot{\eta} = -m\eta, \quad \dot{\zeta}_i = 0, \quad i = 1, \ldots, l.$$

4.3   Applications: geodesic flows on surfaces with singular metrics

We start with some general consequences of the results in two previous sections and then apply them to several concrete problems connected with singularities of divide-by-zero type.

Let W be a vector field of the type in (4.2), where $r \neq 0, 1$ and the smooth[2] vector field V has the form (4.9). Assume that conditions (4.3) and (4.4) hold true. Let 0 be a singular point of V such that the linearization of V at 0 has at least one non-zero real eigenvalue, i.e., the spectrum is $(\lambda_1, \lambda_2, 0, \ldots, 0)$, where $\lambda_1 \in \mathbb{R} \setminus \{0\}$.

By Theorem 4.3, we have equality (4.8), which in this case reads $\lambda_1 + \lambda_2 = r\lambda_j$, where $j = 1, 2$, or $\lambda_1 + \lambda_2 = 0$. Each of these equalities defines the spectrum of V up to a common factor σ, i.e., it uniquely defines the spectrum of the corresponding direction field. In both cases $\lambda_{1,2} \in \mathbb{R} \setminus \{0\}$, hence in a neighborhood of 0 the set of singular points of V is the center manifold W^c, codim $W^c = 2$.

Theorem 4.12. *Assume $W^c \subset \Gamma$, then in a neighborhood of 0 the following statements hold.*

(i) *There exists a smooth regular function $g : \Gamma \to \mathbb{R}$ such that $W^c = \{g = 0\}$ and $V|_\Gamma = gV_1|_\Gamma$, where $V_1|_\Gamma$ is a smooth non-vanishing field on Γ.*

(ii) *The spectrum of the linearization of V at any singular point is $\sigma(1, r-1, 0, \ldots, 0)$, where σ is a smooth non-vanishing function on W^c.*

(iii) *The field V is smoothly orbitally equivalent to one of the following normal forms:*

- *(4.20) with $\lambda(z) = r - 1$ if $r > 1$ and $r - 1, (r - 1)^{-1} \notin \mathbb{N}$ or $r < 1$ and $r \notin \mathbb{Q}$,*
- *(4.21) with $\lambda(z) = n$ if $r - 1$ or $(r - 1)^{-1}$ is equal to $n \in \mathbb{N}$,*
- *(4.27) if $r - 1 = -n/m$, where $n, m \in \mathbb{N}$, and $\Psi_i(0,0) \neq 0$ for at least one index $i = 1, \ldots, l$ in the preliminary form (4.23).*

Proof. For the first statement choose local coordinates (ξ, η, ζ) such that the invariant hypersurface Γ is the hyperplane $\{\xi = 0\}$ and the center manifold W^c is the subspace $\{\xi = \eta = 0\}$. Then the field V has the form

$$\dot{\xi} = \xi v, \quad \dot{\eta} = \xi w_1 + \eta w_2, \quad \dot{\zeta}_i = \alpha_i \xi v + \beta_i(\xi w_1 + \eta w_2), \quad i = 1, \ldots, l,$$

---

[2]For simplicity, we always assume that smooth means C^∞-smooth.

where $v, w_{1,2}$ and α_i, β_i are smooth functions of ξ, η, ζ, and $\lambda_1 = v(0)$, $\lambda_2 = w(0)$. Substituting $\xi = 0$, the field $V|_\Gamma$ has the form $\dot{\eta} = \eta w_2$, $\dot{\zeta}_i = \eta \beta_i w_2$, $i = 1, \ldots, l$. Setting the function $g = \eta w_2$, we get $\Gamma \cap \{g = 0\} = W^c$ and $V|_\Gamma = g V_1|_\Gamma$, where the field $V_1|_\Gamma$ has the form $\dot{\eta} = 1$, $\dot{\zeta}_i = \beta_i$, $i = 1, \ldots, l$.

As for the second statement, according to previous reasonings, at any singular point in a neighborhood of 0 we have equality $\lambda_1 + \lambda_2 = r\lambda_j$, where $j = 1, 2$, or $\lambda_1 + \lambda_2 = 0$. From the hypothesis $W^c \subset \Gamma$ it follows equality (4.8) with $\lambda_j = 0$ is impossible. Indeed, by Theorem 4.3 the spectrum of the linearization of the restriction $V|_\Gamma$ at 0 is $(\lambda_1, \lambda_2, 0, \ldots, 0)$, where the number of zero eigenvalues is less by 1 than in the spectrum of V, i.e., is equal to $l-1$. On the other hand, the inclusion $W^c \subset \Gamma$ implies that the spectrum of the linearization of the restriction $V|_\Gamma$ contains l zeros. Hence we have equality $\lambda_1 + \lambda_2 = r\lambda_j$, with $j = 1$ or 2. Without loss of generality one can put $j = 1$, then $\lambda_2 \equiv (r - 1)\lambda_1$. Since the last equality holds identically at all points in W^c, the spectrum is $\sigma(1, r - 1, 0, \ldots, 0)$ with a smooth non-vanishing function σ. The third statement follows from Theorems 4.8–4.11 and Remark 4.1. ∎

Each of the applications in this section will cast in the following situation. Consider the Euler–Lagrange equation

$$\frac{d}{dt}L_p - L_x = 0, \quad p = \frac{dx}{dt}, \tag{4.30}$$

with Lagrangian $L(t, x, p)$, where $t, x \in \mathbb{R}$. In the (t, x, p)-space equation (4.30) generates the direction field χ corresponding to the vector field W given by

$$\dot{t} = L_{pp}, \quad \dot{x} = pL_{pp}, \quad \dot{p} = L_x - L_{tp} - pL_{xp}, \tag{4.31}$$

where the dot over a symbol means differentiation with respect to an independent variable playing the role of time.

Lemma 4.13. *At all points of the (t, x, p)-space where L is smooth the identity $D_W \equiv 0$ holds. Consequently, at all singular points of the vector field W where L is smooth, the spectrum of the linearization of W has resonance $\lambda_1 + \lambda_2 = 0$. The same statements are valid for the corresponding direction field χ.*

Proof. The identity $D_W \equiv 0$ is due to simple calculation. The field (4.31) belongs to the class of vector fields of type (4.9), where the generators of the ideal I are $v = L_{pp}$ and $w = L_x - L_{tp} - pL_{xp}$. Hence the spectrum of the linearization of W at any singular point is $(\lambda_1, \lambda_2, 0)$. The equality $\lambda_1 + \lambda_2 = 0$ for the field W follows from the equality $D_W \equiv 0$. The same equality for the fields φW follows from the identity $D_{\varphi W} \equiv \varphi D_W + L_W \varphi$. ∎

In the applications below we deal with the case when the Lagrangian is smooth at all points of the (t, x, p)-space except for the the points of some regular surface $\Gamma = \{f = 0\}$ and the components of the field W given by formula (4.31) are fractions with common denominator f^r, $r > 0$. Thus the field W is connected with some smooth field V by the formula (4.2). From the identity $D_W \equiv 0$ (Lemma 4.13) it follows that conditions (4.3) and (4.4) will be always satisfied, hence Theorems 4.1–4.4 and 4.12 are valid.

4.3.1 Pseudo-Riemannian metrics

Consider a surface S with a system of coordinates (t, x) and a pseudo-Riemannian metric

$$Q(dt, dx) = a(t, x)\, dx^2 + 2b(t, x)\, dx dt + c(t, x)\, dt^2 \qquad (4.32)$$

with smooth coefficients a, b, c. The quadratic form Q is positive definite on an open domain $\mathcal{E} \subset S$ (which is called *elliptic*), indefinite on some other open domain $\mathcal{H} \subset S$ (which is called *hyperbolic*), and degenerate on the curve $A = \{\Delta = 0\}$, where $\Delta = b^2 - ac$ is the discriminant of the form Q. The curve A separates the domains $\mathcal{E}$ and $\mathcal{H}$, every point of A is said *parabolic*.

Example 4.5. Let S be a smooth surface embedded in the 3-dimensional Minkowski space, i.e., the 3D affine space with Cartesian coordinates (x, y, z) endowed with the pseudo-Euclidean metric $ds^2 = dx^2 + dy^2 - dz^2$. A pseudo-Riemannian metric is induced on S by the metric ds^2 in the ambient space. Denote by C_P the *light cone* in the 3D tangent space at the point $P = (x, y, z)$ given by the equation $dx^2 + dy^2 - dz^2 = 0$. Then three possibilities arise: either the tangent plane to S at P does not intersect C_P (then $P \in \mathcal{E}$), or it intersects C_P along a pair of lines (then $P \in \mathcal{H}$), or finally it intersects C_P along a unique line (then P is parabolic).

For instance, if S is a Euclidean sphere ($x^2 + y^2 + z^2 = r^2$), the parabolic points form two circles $z = \pm r/\sqrt{2}$, which separate S into two elliptic domains ($\mathcal{E} : |z| > r/\sqrt{2}$) and one hyperbolic domain ($\mathcal{H} : |z| < r/\sqrt{2}$). Geodesics on Euclidean spheres and ellipsoids in 3D Minkowski space are well-studied, see e.g. [24], [31].

Consider geodesics generated by the pseudo-Riemannian metric (4.32) in a neighbourhood of a parabolic point. Their 1-graphs are extremals of equation (4.30) with $L = \sqrt{F}$, where $F = a(t, x)p^2 + 2b(t, x)p + c(t, x)$. Then the vector field W given by formula (4.31) reads

$$\dot{t} = -\Delta F^{-\frac{3}{2}}, \quad \dot{x} = -p\Delta F^{-\frac{3}{2}}, \quad \dot{p} = -MF^{-\frac{3}{2}}/2, \qquad (4.33)$$

where $M = \sum_{i=0}^{3} \mu_i(t, x)p^i$ is a cubic polynomial in p with coefficients

$$\mu_3 = a(a_t - 2b_x) + ba_x, \quad \mu_2 = b(3a_t - 2b_x) + ca_x - 2ac_x,$$
$$\mu_1 = b(2b_t - 3c_x) + 2ca_t - ac_t, \quad \mu_0 = c(2b_t - c_x) - bc_t.$$

Multiplying W by $-F^{\frac{3}{2}}$, we obtain the field V

$$\dot{t} = \Delta, \quad \dot{x} = p\Delta, \quad \dot{p} = M/2. \qquad (4.34)$$

For any point $q_* = (t_*, x_*) \in \mathcal{E} \cup \mathcal{H}$ and any $p \in \mathbb{R}P$ there exists a unique geodesic passing through q_* with given tangential direction p. However if q_* is parabolic, this is not the case. Indeed, for any tangential direction $p \in \mathbb{R}P$ such that $M(q_*, p) \neq 0$ there exists a unique trajectory of (4.34) passing through the point (q_*, p), a vertical line, which projects onto the single point q_* in the (t, x)-plane. Thus, geodesics outgoing from q_* must have tangential directions p such that $M(q_*, p) = 0$, i.e., their 1-graphs pass through a singular point (q_*, p) of the field V.

Let $q_* \in A$ and consider the equation $M(q_*, p) = 0$ with respect to p. We shall assume that in a neighborhood of q_* the curve A is regular and $a(q_*) \neq 0$. Then the quadratic polynomial $F(q_*, p) = ap^2 + 2bp + c$ has a unique root $p_0(q_*) = -\frac{b(q_*)}{a(q_*)}$, that is, the isotropic direction.[3] A simple substitution shows that p_0 is a root of the cubic polynomial $M(q_*, p)$. Assume that the isotropic direction p_0 is not tangent to the curve A at q_*, i.e, $(a\Delta_t - b\Delta_x)|_{q_*} \neq 0$.

Under the assumptions above, the cubic polynomial $M(q_*, p)$ has one or three real prime roots.[4] Define $W_0^c = \{q \in A, p = p_0(q)\}$ and $W_{\pm}^c = \{q \in A, p = p_{\pm}(q)\}$ where $p_{\pm}(q_*)$ are the non-isotropic roots of $M(q_*, p) = 0$, if they exist. The union of the three curves W_0^c, $W_{\pm}^c$ is the set of singular points of V and coincides with its center manifold W^c. The function F vanishes on W_0^c while $F \neq 0$ on $W_{\pm}^c$. Thus the fields (4.33) and (4.34) are connected by relation (4.2), where $f = F$ and $r = \frac{3}{2}$. Since the field W is obtained from an Euler–Lagrange equation, conditions (4.3) and (4.4) follow from the identity $D_W \equiv 0$, which is valid for all points except for the hypersurface $\Gamma = \{F = 0\}$. From Theorem 4.1 it follows that Γ is an invariant hypersurface of V. Hence the isotropic curves are geodesic lines (of zero length) in the pseudo-Riemannian metric (4.32). By construction $W_0^c \subset \Gamma$.

Let $(q, p_0) \in W_0^c$. Clearly, the spectrum of the linearization of V at (q, p_0) contains the eigenvalue $\lambda_1 = \Delta_t + p_0\Delta_x \neq 0$. By Theorem 4.12, in a neighborhood of (q, p_0) there exists a function $\sigma : W_0^c \to \mathbb{R}$ such that the spectrum of the linearization of V at all points sufficiently close to (q, p_0) is $\sigma(2, 1, 0)$. Computing, we easily get $\sigma = \Delta_t + p_0\Delta_x$. Hence the germ of V at (q, p_0) is smoothly orbitally equivalent to

$$\dot{\xi} = 2\xi + \varphi(\zeta)\eta^2, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0. \tag{4.35}$$

The normal form (4.35) can be further simplified.

Theorem 4.14. *The germs of the vector field V given by formula (4.34) at the singular points $(q, p_0) \in W_0^c$ and $(q, p_{\pm}) \in W_{\pm}^c$ are smoothly orbitally equivalent to*

$$\dot{\xi} = 2\xi, \quad \dot{\eta} = \eta, \qquad \dot{\zeta} = 0, \tag{4.36}$$

$$\dot{\xi} = \xi, \qquad \dot{\eta} = -\eta, \quad \dot{\zeta} = \xi\eta, \tag{4.37}$$

respectively.

Proof. To establish normal form (4.36) it is sufficient to prove that the coefficient $\varphi(\zeta)$ in the normal form (4.35) is identically equal to zero. Indeed, the field (4.35) has the invariant foliation $\{(\xi, \eta, \zeta) : \zeta = c\}$, and the restriction to each leaf is a node with exponent[5] equal to 2. The eigenvalue of largest modulus corresponds to the eigenvector $\frac{\partial}{\partial \xi}$ and the eigenvalue of smallest modulus corresponds to $\frac{\partial}{\partial \eta}$.

Given an arbitrary point $(q_*, p_0) \in W_0^c$, consider the restriction of the field (4.35) to the invariant leaf $\{(\xi, \eta, \zeta) : \zeta = \zeta_*\}$ passing through (q_*, p_0). Integrating

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[3]The light cone at a parabolic point consists of a unique isotropic line.

[4]If S is a surface embedded in 3D Minkowski space, these cases correspond to positive or negative Gaussian curvature of S in the Euclidean metric $dx^2 + dy^2 + dz^2$.

[5]The exponent of a node (or saddle) is defined to be the ratio of the eigenvalue of largest modulus of the linearization field to the smallest one.

the corresponding differential equation $d\xi/d\eta = 2\xi/\eta + \varphi(\zeta_*)\eta$, we get the single integral curve $\eta = 0$ and the family of integral curves

$$\xi = c\eta^2 + \varphi(\zeta_*)\eta^2 \ln|\eta|, \quad c = \text{const}, \tag{4.38}$$

with common tangential direction $\frac{\partial}{\partial \eta}$ at 0.

In the case $\varphi(\zeta_*) = 0$ all curves of the family (4.38) are parabolas, in the case $\varphi(\zeta_*) \neq 0$ they are C^1-smooth, but not C^2-smooth at 0. On the other hand, the previous reasoning shows that the germ of V at (q_*, p_0) has at least one C^∞-smooth integral curve: the vertical line (parallel to the p-axis). Simple calculation shows that the direction $\frac{\partial}{\partial p}$ in the initial coordinates (t, x, p) corresponds to the direction $\frac{\partial}{\partial \eta}$ in the normal coordinates (ξ, η, ζ). Hence family (4.38) contains at least one C^∞-smooth integral curve. This implies that $\varphi(\zeta_*) = 0$.

The second statement of the theorem (the normal form (4.37)) follows from Lemma 4.13 and Theorem 4.11; validity of the condition $\Psi_1(0, 0) \neq 0$ can be proved by direct calculation (see Theorem 2 in [38]). ∎

4.3.2   Metrics of Klein type

A natural generalization of the Klein metric on the (t, x)-plane is

$$ds^2 = \frac{\alpha \, dx^2 + 2\beta \, dx dt + \gamma \, dt^2}{t^{2n}}, \quad n \in \mathbb{N}, \tag{4.39}$$

where the numerator is a positive definite quadratic form[6] with coefficients α, β, γ smoothly depending on t, x. We study locally the geodesics of metric (4.39) passing through a singular point, i.e., a point of the axis $\{t = 0\}$. It is not hard to prove that in appropriate local coordinates on the (t, x)-plane the germ of metric (4.39) simplifies to the form

$$ds^2 = \frac{\alpha \, dx^2 + \gamma \, dt^2}{t^{2n}}, \quad n \in \mathbb{N}, \tag{4.40}$$

with smooth positive coefficients $\alpha(t, x)$ and $\gamma(t, x)$.

The geodesics of metric (4.40) are extremals of the Euler–Lagrange equation (4.30) with $L = \sqrt{F}/t^n$, where $F = \alpha p^2 + \gamma > 0$ and $p = dx/dt$. The corresponding vector field W reads

$$\dot{t} = \alpha\gamma t^{-n} F^{-\frac{3}{2}}, \quad \dot{y} = \alpha\gamma p t^{-n} F^{-\frac{3}{2}}, \quad \dot{p} = -\frac{1}{2} t^{-n-1} M F^{-\frac{3}{2}}, \tag{4.41}$$

where $M = \sum\limits_{i=0}^{3} \mu_i(t, x) p^i$ is a cubic polynomial of p with coefficients

$$\mu_3 = \alpha(t\alpha_t - 2n\alpha), \quad \mu_2 = t(\alpha_x\gamma - 2\alpha\gamma_x), \quad \mu_1 = t(2\alpha_t\gamma - \alpha\gamma_t) - 2n\alpha\gamma, \quad \mu_0 = -t\gamma\gamma_x.$$

Multiplying W by f^{n+1}, where $f = t g^{\frac{1}{n+1}}$ and $g = F^{3/2}/(\alpha\gamma) > 0$, we obtain the field V

$$\dot{t} = t, \quad \dot{x} = pt, \quad \dot{p} = -M/(2\alpha\gamma). \tag{4.42}$$

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[6]The case when the numerator is an indefinite (and non-degenerate) quadratic form was also studied [39], but for our present purposes it is sufficient to consider the positive definite case.

Fields (4.41) and (4.42) are connected by relation (4.2), where $f = tg^{\frac{1}{n+1}}$ and $r = n + 1$. Conditions (4.3) and (4.4) are satisfied, W being obtained from an Euler–Lagrange equation. Theorem 4.1 implies that $\Gamma = \{f = 0\} = \{t = 0\}$ is an invariant plane for V. The restriction of the field V to Γ is parallel to the p-axis. Hence geodesics outgoing from a point $q_* = (0, x_*)$ must have tangential directions corresponding to p such that $M(q_*, p) = 0$, i.e., their 1-graphs pass through singular points of V.

Given a point $q_* = (0, x_*)$, consider the equation $M(q_*, p) = 0$ with respect to p. Since $M(q_*, p) = -2n\alpha p(\alpha p^2 + \gamma)$, the cubic polynomial $M(q_*, p)$ has the only real root $p = 0$. The spectrum of the linearization of V at $(q_*, 0)$ is $(n, 1, 0)$, and the x-axis is the center manifold (W^c). Clearly, $W^c \subset \Gamma$ and from Theorem 4.12 we get the following result.

Theorem 4.15. *The germ of the vector field (4.42) at the singular point $(q_*, 0)$ is smoothly orbitally equivalent to*

$$\dot{\xi} = n\xi + \varphi(\zeta)\eta^n, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0. \tag{4.43}$$

Unlike the case of geodesics in pseudo-Riemannian metrics, here the coefficient $\varphi(\zeta)$ is not necessarily zero. For instance, in the case $n = 1$ the condition $\varphi(\zeta_*) = 0$ is equivalent to $\gamma_x(0, x_*) = 0$, where ζ_* corresponds to the point $q_* = (0, x_*)$. Clearly, if $\varphi(0) \neq 0$ then $\varphi(\zeta)$ simplifies to 1, if $\varphi(\zeta)$ has a finite order s at the origin then $\varphi(\zeta)$ simplifies to ζ^s.

Example 4.6. Consider the Klein metric, given by formula (4.40) with $\alpha \equiv \gamma \equiv 1$ and $n = 1$. In this case vector field V given by (4.42) has the normal form (4.43) with $\varphi(\zeta) \equiv 0$, since $\gamma_x(0, x) \equiv 0$. Hence the restriction of the field V on each invariant leaf (given by the formula $\zeta = c$ in the normal coordinates) is a bicritical node. Thus the integral curves of V are C^∞-smooth, and for each singular point $q_* = (0, x_*)$ there exists a family of geodesics with common tangential directions $p = 0$ and various 2-jets. Indeed, geodesics of the Klein metric passing through the point $q_* \in A$ (here A is the absolute) are the circles $(x - x_*)^2 + t^2 = R^2$ and the straight line $x = x_*$.

4.3.3 Almost-Riemannian metrics

In this section we study geodesics starting at a Grushin point of a generic 2-dimensional almost-Riemannian structure. According to Theorem 1.3, the problem is equivalent to studying geodesics starting at $(0, 0)$ for the almost-Riemannian structure on $\mathbb{R}^2$ whose local orthonormal frame is given by $V_1 = \frac{\partial}{\partial x}$, $V_2 = xe^{\phi(x,y)}\frac{\partial}{\partial y}$. Let $v(x, y) = 2e^{-\phi(x,y)}$. Then the orthonormal frame takes the form

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = 2xv^{-1}(x, y)\frac{\partial}{\partial y}, \tag{4.44}$$

where $v(x, y)$ is a smooth non-vanishing function. The almost-Riemannian metric is

$$ds^2 = dx^2 + \frac{v^2}{4x^2}\,dy^2 = \frac{v^2\,dy^2 + d(x^2)^2}{4x^2}.$$

Substituting $t = x^2$ and multiplying by the constant factor 4, we get

$$ds^2 = \frac{\widetilde{v}^2\,dy^2 + dt^2}{t}, \quad \widetilde{v}(t,y) = v_1(t,y) + \sqrt{t}\,v_2(t,y), \tag{4.45}$$

where $v_{1,2}$ are smooth functions defined by decomposition of the function $v(x,y)$ into even and odd parts: $v(x,y) = v_1(x^2,y) + xv_2(x^2,y)$. Geodesics of the metric (4.45) are extremals of Euler–Lagrange equation with Lagrangian $L = \sqrt{F/t}$, where $F = \widetilde{v}^2 p^2 + 1$ and $p = dy/dt$. The corresponding vector field W in the (t,y,p)-space reads

$$\dot{t} = \widetilde{v}^2 t^{-\frac{1}{2}} F^{-\frac{3}{2}}, \quad \dot{y} = \widetilde{v}^2 p t^{-\frac{1}{2}} F^{-\frac{3}{2}}, \quad \dot{p} = \frac{\widetilde{v}}{2} t^{-\frac{3}{2}} F^{-\frac{3}{2}} \widetilde{M}, \tag{4.46}$$

where $\widetilde{M} = \sum\limits_{i=0}^{3} \widetilde{\mu}_i(t,y)p^i$ is a cubic polynomial in p with coefficients

$$\widetilde{\mu}_3 = \widetilde{v}^3 - 2t\widetilde{v}^2\widetilde{v}_t, \quad \widetilde{\mu}_2 = -2t\widetilde{v}_y, \quad \widetilde{\mu}_1 = \widetilde{v} - 4t\widetilde{v}_t, \quad \widetilde{\mu}_0 = 0. \tag{4.47}$$

Multiplying W by f^r, where $r = \frac{3}{2}$, $f = tg$ and $g = (2/\widetilde{v})^{\frac{2}{3}} F \neq 0$, we obtain the field V

$$\dot{t} = 2\widetilde{v}t, \quad \dot{y} = 2\widetilde{v}tp, \quad \dot{p} = \widetilde{M}. \tag{4.48}$$

Fields (4.46) and (4.48) are connected by relation (4.2), where the function $f = tg$ is regular and $r = \frac{3}{2}$. Nevertheless, in general we cannot apply Theorems 4.1, 4.3 and 4.12, since the field V is not even C^1-smooth. Indeed, the components of the field V depend on the function $\widetilde{v}(t,y)$ and its first-order derivatives, which are smooth only if $v(x,y)$ is an even function of x (see formula (4.45)).

Example 4.7. Consider the Clairaut–Liouville metric. This is an example in which the vector field (4.48) turns out to be smooth, the function $v(x,y)$ being even in x. For instance, in [10] the authors deal with the metric

$$ds^2 = dx^2 + \frac{g(x^2,y)}{x^2}\,dy^2 = \frac{x^2\,dx^2 + g(x^2,y)\,dy^2}{x^2},$$

where g is a positive smooth function (x and y are standard angular coordinates on the sphere, the curve $A = \{x = 0\}$ is the equator).[7] After the change of variables $t = x^2$ we get the metric (4.45) with $\widetilde{v} = 2\sqrt{g(t,y)}$, which leads to the smooth field (4.48).

To overcome the problem, we make the change of variable $x^2 = t$ in (4.48). This yields to

$$\dot{x} = xv, \quad \dot{y} = 2x^2 vp, \quad \dot{p} = M, \tag{4.49}$$

where $M = \sum\limits_{i=0}^{3} \mu_i(x,y)\,p^i$ and $\mu_i(x,y) = \widetilde{\mu}_i(x^2,y)$. The coefficients $\widetilde{\mu}_i$ are polynomials of the function $\widetilde{v}(t,y)$ and its first-order derivatives (see formulas (4.47)). Note also that $\widetilde{v}_t$ appear in (4.47) with the factor t, whence after the substitution $x^2 = t$ the expression $t\widetilde{v}_t$ becomes a smooth function of x,y.

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[7]In the case $g(x^2,y) \equiv 1$ this formula gives the well-known Grushin metric.

The first two components of the field (4.49) vanish at $x = 0$. Given a point $q = (0, y)$ consider the cubic equation $M(q, p) = 0$ with respect to p. It reads $v(q)p((v(q)p)^2 + 1) = 0$. This equation has a unique real root $p_0 = 0$. Recalling that $p = dy/dt$, the root $p_0 = 0$ defines the unique admissible direction for geodesics passing through the point $(0, y)$ on the (t, y)-plane. The corresponding direction on the (x, y)-plane is given by the relation $dy/dx = 2xp$ which is also equal to zero. The spectrum of the linearization of the germ (4.49) at (q, p_0) is $(\lambda_1, \lambda_2, 0)$, where $\lambda_1 = v(q)$ and $\lambda_2 = M_p(q, p_0) = v(q)$.

Theorem 4.16. *The germ of the vector field (4.49) at the singular point (q, p_0) is smoothly orbitally equivalent to*

$$\dot{\xi} = \xi, \quad \dot{\eta} = \eta, \quad \dot{\zeta} = 0. \tag{4.50}$$

Proof. By Theorem 4.9, the germ of the vector field (4.49) at (q, p_0) is smoothly orbitally equivalent to normal form (4.21) with $\lambda(\zeta) \equiv 1$ and $l = 1$. To establish normal form (4.50) it is sufficient to prove that the coefficient $\varphi(\zeta)$ in (4.21) is identically equal to zero.

Let Λ be the linearization of the vector field (4.49) at the singular point (q, p_0). Consider the matrix $\Lambda - \lambda I$, where $\lambda = v(q)$ is the double eigenvalue of Λ. Clearly, the value $\mathrm{rank}(\Lambda - \lambda I)$ equals either 1 or 2 and it is an invariant of the field. Hence $\varphi(0) = 0$ if $\mathrm{rank}(\Lambda - \lambda I) = 1$ and $\varphi(0) \neq 0$ if $\mathrm{rank}(\Lambda - \lambda I) = 2$. On the other hand, a simple calculation shows that $\mathrm{rank}(\Lambda - \lambda I) = 1$ if and only if $M_x(q, p_0) = 0$. Recalling that $M(q, p) = v(q)p((v(q)p)^2 + 1)$ and $p_0 = 0$ we get $M_x(q, p_0) = 0$. This completes the proof. ∎

From the normal form (4.50) it follows that vector field (4.49) has an invariant foliation (given by $\zeta = \mathrm{const}$ in the normal coordinates) such that each leaf intersects the center manifold W^c at a unique point. Hence, geodesics passing through the point $q_* = (0, y_*)$ on the (x, y)-plane are projections of integral curves lying in the corresponding leaf. The restriction of vector field (4.49) to the leaf is a bicritical node, hence there is a one-parameter family of integral curves passing through the point $q_* = (0, y_*)$. This gives a family of smooth geodesics passing through the point q_* with common tangential direction which coincides with V_1 and V_2 at the point q_*. Moreover, the geodesics have the same 2-jet and different 3-jets at q_*.

Example 4.8. Geodesics in the Grushin metric (which corresponds to the vector fields (4.44) with $v(x, y) \equiv 2$) have the form $y(x) = y_* + c^{-2} \arcsin(cx) - c^{-1}x\sqrt{1 - c^2x^2}$, where c is an arbitrary constant.

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