Regularization of chattering phenomena via bounded variation controls

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Abstract

In optimal control there may be bad oscillatory phenomena. For instance, this is the case for Fuller's phenomenon [9] where the optimal control is bang-bang with an infinite number of switchings in finite time. In the framework of hybrid systems, there are optimization problems where the discrete part of the dynamics switches infinitely many times in finite time [13] (and this is known as Zeno's phenomenon.)

In this paper we provide a technique to regularize this kind of phenomena. Namely, we define a BV regularization of a general optimal control problem and show that the solution of the perturbed problem is quasi-optimal for the reference problem in the sense that the loss of optimality is small. We apply our results to hybrid systems and we estimate the decay of the error, when the total variation of the control grows.

1 Introduction

Chattering phenomena in optimal control have been known since the first example presented in [9]. Roughly speaking, chattering is a degeneracy phenomenon in which a control oscillates or switches infinitely many times over a finite time interval. To explain this behavior, let us recall the example in [9], also known as *Fuller's problem.* Consider the control system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \tag{1}$$

where an external agent acts on the system by modifying the acceleration, and the control $u : [0, T] \to [-1, 1]$ is measurable. The optimal control problem is to minimize the functional

$$\int_0^T x_1^2(t)\, dt \tag{2}$$

among trajectories of (1) steering an initial point (x_1^0, x_2^0) to the origin, i.e., satisfying initial and terminal constraints

$$\begin{gathered} x_1(0) = x_1^0, \quad x_2(0) = x_2^0, \\ x_1(T) = 0, \quad x_2(T) = 0. \end{gathered} \tag{3}$$

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It has been shown that for every (x_1^0, x_2^0) there exists a unique control $u^0 : [0, T^0] \to [-1, 1]$ which is a solution to (1), (2), (3) and which has the following form

$$u(t) = \begin{cases} 1, & t \in [t_{2k}, t_{2k+1}], k \in \mathbb{N} \\ -1, & t \in [t_{2k+1}, t_{2k+2}], k \in \mathbb{N} \end{cases}$$

where $\{t_k\}_{k\in\mathbb{N}}$ is an increasing sequence of switching times, depending on the initial condition (x_1^0, x_2^0) and converging to $T^0 < \infty$. Although at first sight one could think this kind of degeneracy is related to specific symmetries of the system, it turns out that this behavior is rather typical. Indeed, it was later shown in the fundamental work [14] that the set of single-input optimal control problems which have control-affine Hamiltonian and whose solution is chattering is an open semi-algebraic set.

Optimal controls showing this degenerate structure has been found for a variety of problems: besides the ones mentioned above, this phenomenon also concerns state constrained problems and hybrid systems. For instance, in [16] the author studied an optimal control problem with an inequality state constraint of third order and showed that the optimal trajectory touches the constraint boundary at an infinite sequence of times approaching a finite limit (Robbins' phenomenon). As it happens for Fuller's phenomenon, it turns out that Robbins' phenomenon is in a sense generic, when the order of the state constraint is sufficiently high, see [4]. In the framework of hybrid systems, chattering is also known as Zeno's phenomenon and it is related to the presence of trajectories whose discrete part jumps infinitely many times in a finite time interval (see the two examples in [11]).

Chattering causes several difficulties in theoretical aspects of optimal control as well as in applications. From a theoretical point of view it prevents a direct application of Pontryagin's Maximum Principle because of the lack of a positive length interval where the control function is continuous. This implies that, when chattering is included in the analysis, finding necessary and sufficient optimality conditions becomes more intricate (see [8] for the case of state constrained problems). Some results in this sense were proved in [18], yet the problem is not completely understood in many contexts, such as state constrained problems or hybrid systems [13]. Another delicate issue comes from the study of regularity properties of optimal syntheses [5, 15].

As concerns applications, chattering phenomena are often an obstacle when using numerical methods to attack optimal control problems. For instance, for single-input problems with a scalar state constraint the presence of chattering may imply ill-posedness (non-invertible Jacobian) of shooting methods [3]. Likewise, as remarked by the authors in [1], the interior point-based algorithm developed appears not to converge when applied to Fuller's problem.

The main motivation of this paper is to provide convergence of numerical methods when chattering takes place. We consider a fully nonlinear control system and optimize a cost of Bolza type with continuous Lagrangian. We define a sequence of relaxed problems obtained by adding a term of total variation of the control in the cost functional. Thanks to small time local controllability, under equiboundedness of trajectories and some convexity properties of the set of velocities for the augmented system, we show the convergence of the cost functional along a sequence of solutions of the relaxed problems.

We also study how fast the cost converges as the total variation of controls in the approximating sequence grows. Namely, we estimate explicitly the rate of convergence of the cost along suboptimal regimes obtained by suitable truncations of the chattering one in terms of switching times. A related result was proved in [19, 20] for small perturbations of the Fuller's problem. In those papers, the authors exhibit a sequence of suboptimal regimes for a control system having (1) as principal part and for the cost (2) and they prove that the cost converges with the same rate as the sequence of switching times (of the chattering control). Here we obtain a similar result for nonlinear control systems with a general cost of integral type, only assuming small time local controllability at the final point and holderianity of the time-optimal map. In particular, this allows us to sharpen the convergence rate of the cost along the sequence of solutions of the relaxed problem.

Notice that for the case considered in [20], the rate of convergence happens to be exponential as a function of the number of switchings. Likewise, for the class of systems considered in [14], the switching times converge exponentially to the final time. Nevertheless, whether a slower rate of convergence is "typical" still remains an open question.

Finally, we apply our result to optimization problems for hybrid systems, obtaining estimates of the cost convergence as the number of switchings grows.

The paper is organized as follows. In Section 2 we define chattering and state the main results. Applications of the main results to hybrid systems are given in Section 3. Section 4 is devoted to the proofs of Theorems 1 and 2. Finally, we show in Appendix 5 an existence result for optimal control problems without convexity assumptions, using a total variation term in the cost.

2 Main results

Consider the control system

$$\dot{x} = f(x, u), \quad u \in \mathcal{U}, \tag{4}$$

where $f \in \mathcal{C}^\infty(\mathbb{R}^N \times \mathbb{R}^m, \mathbb{R}^N)$ and $f(0,0) = 0$. Given $\mathbf{U} \subset \mathbb{R}^m$, set

$$\mathcal{U} = \{u : [0, t_u] \to \mathbf{U} \text{ measurable}, t_u > 0\}.$$

Denote by

$$\mathcal{F} = \{f(\cdot, u) : \mathbb{R}^N \to \mathbb{R}^N \mid u \in \mathbf{U}\}$$

the family of vector fields associated with the dynamics of (4). Given an initial state x_0, a Lagrangian $L \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^m)$, and $\varepsilon \geqslant 0$ we consider the optimal control problem

$$\min_{u \in \mathcal{U}, T > 0} \left(\int_0^T L(s, x(s), u(s))\, ds + \varepsilon TV(u) \right), \tag{5}$$

with initial and final constraints

$$x(0) = x_0, \quad x(T) = 0, \tag{6}$$

where $TV(u)$ denotes the total variation of u.

Definition 1. *By* chattering *control we mean a measurable $u : [0, t_u] \to \mathbf{U}$ such that there exists an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ converging to t_u with the property that $TV(u|_{[0,t_n]}) < \infty$ for every n, and*

$$\lim_{n \to \infty} TV(u|_{[0,t_n]}) = +\infty.$$

In the following result we consider the case in which problem (4), (5), (6), with $\varepsilon = 0$ has an optimal solution which either has bounded total variation either is chattering in the sense above.

Theorem 1. *Assume that*

(i) $\mathrm{Lie}_0\, \mathcal{F} = T_0\mathbb{R}^N$ *and* 0 *is small time locally controllable for the control system* (4)*;*

(ii) the optimal control problem (4), (5), (6) *with $\varepsilon = 0$ admits a unique solution $u^* : [0, T^*] \to \mathbf{U}$ whose corresponding trajectory is denoted by x^*;*

(iii) for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ the set

$$V(t,x) = \{(f(x,u), L(t,x,u) + \gamma) \mid u \in \mathbf{U}, \gamma \geqslant 0\}$$

is convex;

(iv) there exists $b > 0$ such that, for every $u \in \mathcal{U}$ whose corresponding trajectory satisfies (6), we have

$$t_u + ||x_u||_\infty \leqslant b$$

and $\mathbf{U}$ is compact.

Then, for every $\varepsilon > 0$, the optimal control problem (4), (5), (6), admits a solution $u_\varepsilon : [0, T_\varepsilon] \to \mathbf{U}$. Moreover,

$$\lim_{\varepsilon \to 0} \int_0^{T_\varepsilon} L(t, x_\varepsilon(t), u_\varepsilon(t))\, dt = \int_0^{T^*} L(t, x^*(t), u^*(t))\, dt. \tag{7}$$

Theorem 1 implies that whenever the optimal control u^* is chattering, it can be replaced by a non-chattering one u_ε, paying a price that goes to zero as the total variation of u_ε grows. In other words, u_ε is quasi-optimal for the problem with $\varepsilon = 0$ in the sense that the cost along u_ε converges to the value function of the reference problem. To prove Theorem 1 we use an auxiliary result (see Lemma 2 in Section 4) which provides convergence of the cost along a sequence of controls obtained by truncating the optimal chattering control, without the convexity assumptions *(iii)* or the *a priori* estimates *(iv)* on trajectories.

Remark 1. According to Definition 1, a control having one accumulation of switchings is chattering. When a solution of (4), (5) with $\varepsilon = 0$, (6) presents a finite number of accumulations of switchings the problem can be tackled by applying iteratively Theorem 1.

The estimate on the error as the total variation grows can be sharpened. This is the statement of next theorem which gives the existence of suboptimal controls whose associated cost converges sufficiently fast to the optimal one. The rate of convergence is related to the regularity of the time-optimal map defined below. The idea is to consider the optimal control u^* until a time $T^* - \eta$ and then steer the system to the origin with a control having total variation uniformly bounded with respect to η.

We define the time-optimal map associated with the problem (4), (6) as

$$\Upsilon(x_0) = \inf\{T > 0 \mid \dot{x} = f(x,u), x(0) = x_0, x(T) = 0\}.$$

Theorem 2. *Assume that*

(i) $\mathrm{Lie}_0\, \mathcal{F} = T_0\mathbb{R}^N$ and 0 is small time locally controllable for the control system (4).

(ii) The optimal control problem (4), (5), (6) with $\varepsilon = 0$ admits a unique solution $u^* : [0, T^*] \to \mathbf{U}$ whose corresponding trajectory is denoted by x^*.

(iii) The time-optimal map is $C^{0,\alpha}$ for some $\alpha \in (0,1]$ on a neighborhood of 0.

Then there exist $\eta_0 > 0$ and $C > 0$ such that for any $\eta < \eta_0$ there exists an admissible control $v_\eta : [0, T_\eta] \to \mathbf{U}$ whose corresponding trajectory $x_\eta : [0, T_\eta] \to \mathbb{R}^N$ starting at x_0 satisfies the terminal constraint $x_\eta(T_\eta) = 0$, such that $TV(v_\eta) < \infty$, and

$$\int_0^{T_\eta} L(t, x_\eta, v_\eta)\, dt - \int_0^{T^*} L(t, x^*, u^*)\, dt \leqslant C\eta^\alpha. \tag{8}$$

Moreover, the following convergences hold

$$\lim_{\eta\to 0} T_\eta = T^*$$
$$\lim_{\eta\to 0} \|v_\eta - u^*\|_{L^1} = 0$$
$$\lim_{\eta\to 0} \|x_\eta - x^*\|_\infty = 0.$$

Remark 2. Sufficient conditions for Assumption (iii) have been studied in [2]. Moreover the authors provide an estimate on the Hölder exponent.

The following corollary combines Theorems 1 and 2. It gives an estimate on the rate of convergence of the cost for the optimal control problem (4),(5),(6) with $\varepsilon > 0$ to the optimal cost with $\varepsilon = 0$ in terms of the growth of the total variations of the optimal controls.

Corollary 1. *Let assumptions $(i) - (iv)$ of Theorem 1 hold. Assume moreover that*

(v) *the time-optimal map Υ is $C^{0,\alpha}$ for some $\alpha \in (0,1]$ on a neighborhood of 0;*

Then, for every $\varepsilon > 0$, the optimal control problem (4), (5), (6), admits a solution $u_\varepsilon : [0, T_\varepsilon] \to \mathbf{U}$. Moreover, there exists $M > 0$ such that

$$\int_0^{T_\varepsilon} L(t, x_\varepsilon(t), u_\varepsilon(t))\, dt - \int_0^{T^*} L(t, x^*(t), u^*(t))\, dt \leqslant M(\mu^\alpha + \varepsilon), \tag{9}$$

where μ is such that

$$TV(u^*|_{[0,T^*-\mu]}) \leqslant TV(u_\varepsilon).$$

3 Application to hybrid systems

In this section, we prove a direct adaptation of Theorem 1 to hybrid systems. To this aim, we first recall some basic definitions in the context of hybrid systems (see also [10, 13]).

A *hybrid system* is a collection $\mathcal{H} = (Q, X, f, E, G, R)$ where

- Q is a finite set;
- $X = \{X_q\}_{q\in Q}$ is a collection of subsets $X_q \subset \mathbb{R}^N$ called *locations*;
- $f = \{f_q\}_{q\in Q}$ is a collection of smooth vector fields $f_q : \mathbb{R}^N \to \mathbb{R}^N$;
- $E \subset Q \times Q$ is a subset of edges;
- G is a set-valued map that associates with each edge $(q, q') \in E$ a subset $G(q, q') \subset X_q$ called *guard set*;
- R is a set-valued map that associates with each pair $((q, q'), x) \in E \times X_q$ a subset $R((q, q'), x) \subset X_{q'}$.

A *trajectory* of $\mathcal{H}$ is a triple $(\tau, q(\cdot), x(\cdot))$ where

- $\tau = \{\tau_i\}_{i=0}^M$ is a sequence of increasing positive numbers such that $\tau_0 = 0$ and $M \leqslant \infty$. Set $I = [0, \tau_M]$ if $M < \infty$, $I = [0, \tau_M)$ if $M = \infty$.
- $q : I \to Q$ is such that, for every $i = 0, \dots M - 1$, $q(t)$ is constant on $[\tau_i, \tau_{i+1})$. Set $q_i = q|_{[\tau_i,\tau_{i+1})}$

- for every $i = 0, \ldots, M-1$, $x_i(\cdot) = x|_{(\tau_i, \tau_{i+1})}$ is an absolutely continuous function in (τ_i, τ_{i+1}), continuously prolongable to $[\tau_i, \tau_{i+1}]$ and such that $x_i(t) \in X_{q_i}$.
- for almost every $t \in (\tau_i, \tau_{i+1})$
$$\dot{x}_i = f_{q_i}(x_i). \tag{10}$$
- for every $i = 0, \ldots, M-1$ $(q_i, q_{i+1}) \in E$, $x_i(\tau_{i+1}) \in G(q_i, q_{i+1})$ and, for every $i = 0, \ldots, M-2$, $x_{i+1}(\tau_{i+1}) \in R((q_i, q_{+1}), x_i(\tau_{i+1}))$.

A trajectory $(\tau, q(\cdot), x(\cdot))$ is called *Zeno* if $M = \infty$ and $\tau_\infty < \infty$. Note that in this case we have

$$\tau_\infty = \sum_{i=0}^{\infty} \tau_i < \infty.$$

Given a hybrid system $\mathcal{H}$, a *Lagrangian* for $\mathcal{H}$ is a family $L = \{L_q\}_{q \in Q}$, $L_q : \mathbb{R} \times X_q \to \mathbb{R}$ such that, for every trajectory $(t, q(\cdot), x(\cdot))$ of $\mathcal{H}$ and every $i = 0, \ldots M-1$, the function $t \mapsto L_{q_i}(t, x_i(t))$ is continuous in (t_i, t_{i+1}). Given a Lagrangian for $\mathcal{H}$, we can define the corresponding cost functional C by

$$C(\tau, q(\cdot), x(\cdot)) = \sum_{i=0}^{M-1} \int_{t_i}^{t_{i+1}} L_{q_i}(t, x_i(t))\, dt.$$

Casting Theorem 1 in the language of hybrid systems we obtain the following result. Let $\mathcal{H} = (Q, X, f, E, G, R)$ be a hybrid system with a Lagrangian L and corresponding cost functional C. Fix an initial condition $(q_0, x_0) \in Q \times X_{q_0}$.

Theorem 3. *Let $\mathcal{H}$ be a hybrid system such that* [1] $X_q = \mathbb{R}^N$ *for every* $q \in Q$ *and let L be a Lagrangian for $\mathcal{H}$ with corresponding cost functional $C(\cdot)$. Assume that $(\tau^*, q^*(\cdot), x^*(\cdot))$ is a Zeno trajectory starting at (q_0, x_0) and that*

$$C(\tau^*, q^*(\cdot), x^*(\cdot)) = \min_{(\tau, q(\cdot), x(\cdot))} C(\tau, q(\cdot), x(\cdot)) < \infty,$$

where the minimum is taken over all trajectories of $\mathcal{H}$ starting at (q_0, x_0). Let $\tau^* = \{\tau_i^*\}_{i=0}^{\infty}$. Define the sequence of trajectories $(\tau^n, q^n(\cdot), x^n(\cdot))$ by

- $\tau^n = \{\tau_0^*, \tau_1^*, \ldots, \tau_n^*, \tau_\infty^*\}$;
- $q^n(t) = q^*(t)$ *for every* $t \in [0, \tau_n^*)$, $q^n(t) \equiv q^*(\tau_n^*)$ *for* $t \in [\tau_n, \tau_\infty^*]$;
- $x^n(t) = x^*(t)$ *for every* $t \in [0, \tau_n^*]$ *and for almost every* $t \in [\tau_n^*, \tau_\infty^*]$ x^n *satisfies*
$$\dot{x}^n(t) = f_{q^*(\tau_n^*)}(x^n(t)).$$

Then the following convergences hold

$$||x^n(\cdot) - x^*(\cdot)||_\infty = O(\tau_\infty^* - \tau_n^*) \tag{11}$$
$$C(\tau^n, q^n(\cdot), x^n(\cdot)) - C(\tau^*, q^*(\cdot), x^*(\cdot)) = O(\tau_\infty^* - \tau_n^*). \tag{12}$$

The main idea is to interpret the role of the discrete part of the hybrid system in (10) as a control. Since there are no final constraints the proof simplifies with respect to the proof of Theorems 1, 2.

[1] The assumption $X_q = \mathbb{R}^N$ is not essential and can be dropped, provided that each location X_q is invariant with respect to the vector field $f_{q'}$ for every q'

Proof. Since $q^n(t)$ converges to $q^*(t)$ almost everywhere in $[0,\tau^*_\infty]$, by [17, Theorem 1 p.57] we deduce (11). As for (12), notice that since the Lagrangian is continuous, there exist constants $\tilde{c}, c$ such that $\tilde{c}-c>0$, for every n

$$\int_{\tau_n^*}^{\tau_\infty^*} L_{q^*(\tau_n^*)}(t,x^n(t))dt \leqslant \tilde{c}(\tau_\infty^*-\tau_n^*),$$

and, for every i,

$$L_{q^*(\tau_i^*)}(t,x^*(t)) \geqslant c, \text{almost everywhere in } [\tau_i^*,\tau_{i+1}^*].$$

Therefore,

$$\begin{aligned} 0 \quad &\leqslant \quad C(\tau^n,q^n(\cdot),x^n(\cdot)) - C(\tau^*,q^*(\cdot),x^*(\cdot)) = \int_{\tau_n^*}^{\tau_\infty^*} L_{q^*(\tau_{n_i})}(t,x^n(t))dt - \sum_{i=n}^{\infty}\int_{\tau_i^*}^{\tau_{i+1}^*} L_{q^*(t)}(t,x^*(t))dt \\ &\leqslant \quad \tilde{c}(\tau_\infty^*-\tau_n^*) - c\sum_{i=n}^{\infty}(\tau_{i+1}^*-\tau_i^*) = (\tilde{c}-c)(\tau_\infty^*-\tau_n^*). \end{aligned}$$

This concludes the proof. □

4 Proofs of the main results

4.1 Proof of Theorem 1

Before going into technical details, let us outline the proof of Theorem 1. First, we remark that the local controllability assumption provides existence of a solution $u_\varepsilon, x_\varepsilon$ to (4), (5), (6), for $\varepsilon>0$. Second, we show that, thanks to the assumptions on the velocity sets and the equiboundedness of trajectories, there exists an admissible control w and a positive measurable function γ such that $t\mapsto (f(x_\varepsilon(t),u_\varepsilon(t)), L(t,x_\varepsilon(t),u_\varepsilon(t))$ converges to $t\mapsto (f(x_w(t),w(t)), L(t,x_w(t),w(t))+\gamma(t))$ for the weak* topology in L^∞. Third, we use the optimality of u_ε to prove that w is optimal for the problem with $\varepsilon=0$, which by uniqueness implies $w=u^*$. Fourth, we demonstrate $\gamma=0$ which implies that the Lagrangian cost along u_ε converges to the Lagrangian cost at u^*. This fact is proved thanks to Lemma 2 which exhibits a sequence of admissible controls v_η for which $TV(v_\eta)<\infty$ and the Lagrangian cost converges to the Lagrangian cost at u^*. Finally, to construct v_η we use a topological result, Lemma 1, providing admissible controls steering a point to the origin with total variation uniformly bounded in a neighborhood of the origin.

For the sake of readability, we start by presenting the two auxiliary lemmas mentioned above and then proceed to the proof of the theorem.

Lemma 1. *Assume that* $\mathrm{Lie}_0\,\mathcal{F} = T_0\mathbb{R}^N$, *the origin is small time locally controllable and let* $T_0>0$. *Then there exist* $r, M>0$ *such that for every* y *with* $|y|\leqslant r$ *there exists a piecewise constant control* $w_y:[0,\tau_y]\to\mathbf{U}$ *with the following properties:*

- *(i)* w_y *steers* y *to* 0 *in time* $\tau_y\leqslant T_0$;
- *(ii)* $\sup_{|y|\leqslant r} TV(w_y)\leqslant M$.

Proof. Call $\mathcal{A}(x,[0,T),f)$ the set of points accessible from x in time $t<T$ by trajectories of the control system $\dot{x}=f(x,u), u\in\mathcal{U}$. Since 0 is STLC for the control system (4) defined by f, thanks to [12, Theorem 5.3 a-d], it is also STLC for the system associated with $-f$. Therefore, for every $T>0$ there exists a ball $B_r(0)$ centered in the origin of radius r compactly contained in $\mathcal{A}(0,[0,T),-f)$. Since f is autonomous, for every $y\in B_r(0)$, we have

$$0\in\mathcal{A}(y,[0,T),f) \iff y\in\mathcal{A}(0,[0,T),-f).$$

Since 0 is STLC for $-f$, Theorem 5.3 in [12] implies that 0 is is small time normally self reachable (STNSR) (see [12, Definition 3.6]) for the system associated with $-f$. Applying [12, Theorem 5.5], since $y \in \mathcal{A}(0,[0,T),-f)$ and 0 is STNSR for $-f$ then y is *normally reachable* (see [12, Definition 3.6]) from 0 in time less than T for the system associated with $-f$. Namely, there exist $q = q(y) \in \mathbb{N}$, $u_1, \dots u_q \in \mathbf{U}$ and $(t_1, \dots, t_q)$ with $t_1 + \cdots + t_q \leqslant T$ such that the map

$$(s_1, \dots, s_q) \mapsto E_u(s_1, \dots, s_q) = 0 \circ e^{-s_1 f(\cdot, u_1)} \circ \cdots \circ e^{-s_q f(\cdot, u_q)}, \tag{13}$$

which is defined and smooth in a neighborhood $\mathcal{N}$ of $(t_1, \dots, t_q)$, satisfies $E_u(t_1, \dots, t_q) = y$ and has maximal rank (equal to N) at $(t_1, \dots, t_q)$. Hence there exist a neighborhood $O_y \subset \mathcal{A}(0,[0,T),-f)$ of y on which the map E_u is onto. Moreover, by continuity of the map (13), there exists $c = c(y)$ such that

$$\sup_{x \in O_y} \{s_1 + \cdots + s_q \mid (s_1, \dots, s_q) \in \mathcal{N} \cup E_u^{-1}(x)\} \leqslant c(y)T.$$

In other words every point of O_y is reachable from 0 with piecewise constant control with less than $q = q(y)$ switchings in time smaller than $c(y)T$ via the system $\dot{x} = -f(x,u)$. Reversing time we have that for every point x in O_y there exists a piecewise constant control with less than $q = q(y)$ switchings steering system $\dot{x} = f(x,u)$ from x to 0 in time smaller than $c(y)T$.

By compactness, there exist $y_1, \dots, y_m$ such that $B_r(0) \subset \bigcup_{k=1}^m O_{y_k}$ and, for T sufficiently small, we can assume that $T \sum_{k=1}^m c(y_k) < T_0$. Hence, using the argument above, every point $y \in B_r(0)$ can be steered to 0 in time smaller than $T \sum_{k=1}^m c(y_k)$ by means of a piecewise constant control with less than $\sum_{k=1}^m q(y_k)$ switchings. □

Lemma 2. *Assume that*

(*i*) $\mathrm{Lie}_0 \mathcal{F} = T_0\mathbb{R}^N$ *and* 0 *is small time locally controllable for the control system* (4).

(*ii*) *The optimal control problem* (4), (5), (6) *with* $\varepsilon = 0$ *admits a unique solution* $u^* : [0,T^*] \to \mathbf{U}$ *whose corresponding trajectory is denoted by* x^*.

Then there exists $\eta_0 > 0$ *such that for any* $\eta < \eta_0$ *there exists an admissible control* $v_\eta : [0,T_\eta] \to \mathbf{U}$ *whose corresponding trajectory* $x_\eta : [0,T_\eta] \to \mathbb{R}^N$ *starting at* x_0 *satisfies the terminal constraint* $x_\eta(T_\eta) = 0$, *such that* $TV(v_\eta) \leqslant TV(u^*|_{[0,T^*-\eta]}) + M$, *and*

$$\lim_{\eta \to 0} \int_0^{T_\eta} L(t, x_\eta, v_\eta)\, dt = \int_0^{T^*} L(t, x^*, u^*)\, dt. \tag{14}$$

Moreover, the following convergences hold

$$\lim_{\eta \to 0} T_\eta = T^*$$

$$\lim_{\eta \to 0} \|v_\eta - u^*\|_{L^1} = 0$$

$$\lim_{\eta \to 0} \|x_\eta - x^*\|_\infty = 0.$$

Proof. Let r and M be given by Lemma 1 and let η_0 be such that for every $s \geqslant T^* - \eta_0$ there holds $|x^*(s)| \leqslant r$. For every $\eta < \eta_0$ set $t_\eta = T^* - \eta$. Let $K = \max_{[0,T^*]} |f(x^*(t), u^*(t))|$, then

$$|x^*(t_\eta)| = |x^*(t_\eta) - x^*(T^*)| \leqslant K(T^* - t_\eta) = K\eta.$$

By Lemma 1 there exists a piecewise constant control w_η steering $x^*(t_\eta)$ to 0 in time τ_η with $TV(w_\eta) \leqslant M$. Define v_η by

$$v_\eta(t) = \begin{cases} u^*(t) & \text{for } t \in [0, t_\eta], \\ w_\eta(t - t_\eta) & \text{for } t \in (t_\eta, T_\eta], \\ 0, & \text{for } t \in (T_\eta, \bar{T}], \end{cases} \tag{15}$$

where $T_\eta = t_\eta + \tau_\eta$ and $\bar{T} = \sup_{0 \leqslant \eta < \eta_0} T_\eta$. Denote by $x_\eta(t)$, $t \in [0, \bar{T}]$ the trajectory of (4) associated with the control v_η. By construction

$$TV(v_\eta) \leqslant TV(u^*|_{[0,t_\eta]}) + M.$$

Moreover

$$\lim_{\eta \to 0} T_\eta = T^* \qquad \text{and} \qquad \lim_{\eta \to 0} \|v_\eta - u^*\|_{L^1}.$$

Now set $\mathcal{X}_0 = \{x^*(t) : t \in [0, t_\eta]\} \cup \overline{B_{\eta_0}(0)}$, $C_0 = \sup_{\mathcal{X}_0 \times \mathbf{U}} |f|$, $C_1 = \sup_{\mathcal{X}_0 \times \mathbf{U}} |\partial_x f|$, and $C_2 = \sup_{\mathcal{X}_0 \times \mathbf{U}} |\partial_u f|$. For every $t \in [0, \bar{T}]$ and $\eta < \eta_0$ one has

$$\begin{aligned} |x_\eta(t) - x^*(t)| &= \left| \int_0^t f(x_\eta(s), v_\eta(s)) ds - \int_0^t f(x^*(s), u^*(s)) ds \right| \\ &\leqslant \int_0^{\bar{T}} |f(x_\eta(s), v_\eta(s)) - f(x^*(s), u^*(s))| ds + C_0 |T^* - T_\eta| \\ &\leqslant \int_0^{\bar{T}} |f(x_\eta(s), u^*(s)) - f(x^*(s), u^*(s))| ds + C_2 \|u^* - v_\eta\|_{L^1} + C_0 |T^* - T_\eta| \\ &\leqslant C_1 \int_0^{\bar{T}} |x_\eta(s) - x^*(s)| ds + C_2 \|u^* - v_\eta\|_{L^1} + C_0 |T^* - T_\eta|, \end{aligned}$$

thus, by Gronwall's inequality,

$$\|x_\eta - x^*\|_\infty \leqslant (C_2 \|u^* - v_\eta\|_{L^1} + C_0 |T^* - T_\eta|)\, e^{C_1 \bar{T}}.$$

In particular $\lim_{\eta \to 0} \|x_\eta - x^*\|_\infty = 0$.

Let us show the convergence (14). Notice that, by continuity of L, there exist constants $c \in \mathbb{R}$ and $C > 0$ such that

$$L(t, x^*(t), u^*(t)) \geqslant c, \qquad \text{almost everywhere in } [0, T^*],$$

and

$$|L(t, x, u)| \leqslant \bar{C}, \qquad \text{for almost every } (t, x, u) \in [0, \bar{T}] \times \mathcal{X}_0 \times \mathbf{U}.$$

Then

$$\begin{aligned} 0 &\leqslant \int_0^{T_\eta} L(t, x_\eta(t), v_\eta(t)) dt - \int_0^{T^*} L(t, x^*(t), u^*(t)) dt \\ &= \int_{t_\eta}^{T_\eta} L(t, x_\eta(t), v_\eta(t)) dt - \int_{t_\eta}^{T^*} L(t, x^*(t), u^*(t)) dt \\ &\leqslant \bar{C} \tau_\eta - c(T^* - t_\eta) = \bar{C} \tau_\eta - c\eta \end{aligned} \tag{16}$$

Therefore (14) is proved. □

We are now in a position to prove the main result of the paper.

Proof of Theorem 1. Thanks to assumption (i), there exists a control $u : [0, t_u] \to \mathbf{U}$ steering x_0 to 0 and having bounded variation. Therefore, the existence of a solution $u_\varepsilon : [0, t_\varepsilon] \to \mathbf{U}$ to (4), (5), (6), follows by Theorem 4 in the Appendix. Denoting by x_ε the trajectory corresponding to u_ε we have $x_\varepsilon(0) = x_0$ and $x_\varepsilon(t_\varepsilon) = 0$. Let $v \in \mathcal{U}$ be any admissible control such that $TV(v) < \infty$, and such that the corresponding trajectory satisfies (6). Then, by optimality of u_ε we have

$$\int_0^{t_\varepsilon} L(t, x_\varepsilon, u_\varepsilon)\, dt + \varepsilon TV(u_\varepsilon) \leqslant \int_0^{t_v} L(t, x_v, v)\, dt + \varepsilon TV(v). \tag{17}$$

Thanks to assumptions (iii) and (iv), and since $\mathbf{U}$ is compact, we are going to show that there exists an admissible control $w : [0, t_w] \to \mathbf{U}$ and a positive measurable function $\gamma : [0, t_w] \to \mathbb{R}$ such that the sequence of functions

$$t \mapsto (f(x_\varepsilon(t), u_\varepsilon(t)), L(t, x_\varepsilon(t), u_\varepsilon(t)))$$

converges to

$$t \mapsto (f(x_w(t), w(t)), L(t, x_w(t), w(t)) + \gamma(t))$$

with respect to the weak* topology of L^∞ as ε tends to 0. To see this, consider the augmented system

$$\dot{x} = f(x, u), \quad \dot{x}_{N+1} = L(t, x, u),$$

with constraints

$$x(0) = x_0, \; x_{N+1}(0) = 0, \; x(T) = 0, \; x_{N+1}(T) \in [0, \Gamma],$$

where $\Gamma = \int_0^{t_v} L(s, x_v(s), v(s))\, ds + \varepsilon\, TV(v)$ and v is as above. Denote by $\bar{f}(t, x, u) = (f(x, u), L(t, x, u))$ the augmented dynamics.

Notice that thanks to assumption 2, the sequence t_ε is bounded and converges, up to subsequences, to $t_0 > 0$ as ε tends to 0. Hence, given $\delta > 0$ there exists $\varepsilon_0 > 0$ such that $|t_\varepsilon - t_0| < \delta$ for every $\varepsilon \leqslant \varepsilon_0$ in the chosen subsequence. Since $f(0, 0) = 0$, we can extend $x_\varepsilon, u_\varepsilon$ to $[t_\varepsilon, t_0 + \delta]$ by setting

$$x_\varepsilon(s) \equiv 0, u_\varepsilon(s) \equiv 0, \quad \forall\, s \in [t_\varepsilon, t_0 + \delta].$$

Since the trajectories x_ε are equibounded by assumption (iv), the sequence $s \mapsto \bar{f}(s, x_\varepsilon(s), u_\varepsilon(s))$ is bounded in $L^\infty([0, t_0 + \delta], \mathbb{R}^{N+1})$, whence, up to subsequences, it converges to a function $g \in L^\infty([0, t_0 + \delta], \mathbb{R}^{N+1})$ in the weak* topology. Define

$$\bar{x}(t) = \bar{x}_0 + \int_0^t g(s)\, ds, \quad \bar{x}_0 = (x_0, 0).$$

By construction, $t \mapsto \bar{x}(t)$ is absolutely continuous and, denoting by $\bar{x}_\varepsilon(t) = (x_\varepsilon(t), \int_0^t L(s, x_\varepsilon(s), u_\varepsilon(s)) ds)$,

$$\lim_{\varepsilon \to 0^+} \bar{x}_\varepsilon(t) = \bar{x}(t), \quad \forall\, t \in [0, t_0 + \delta].$$

(Note that by Ascoli Arzelà Theorem, since $\dot{x}_\varepsilon$ is equibounded the above convergence is uniform, i.e., $\bar{x}_\varepsilon$ converges to $\bar{x}$ uniformly on $[0, t_0 + \delta]$.) Let $\bar{x}(t) = (\mathbf{x}(t), \mathbf{x}_{N+1}(t))$. We are going to prove that $\mathbf{x}(t)$ is an admissible trajectory for (4), (6), corresponding to a control w. First notice that by the pointwise convergence above, $\mathbf{x}(\cdot)$ satisfies $\mathbf{x}(t_0) = 0$. Define $\bar{h}_\varepsilon(s) = \bar{f}(s, \mathbf{x}(s), u_\varepsilon(s))$. Then, for every ε, $\bar{h}_\varepsilon$ belongs to the set

$$\mathcal{V} = \{h \in L^2([0, t_0 + \delta], \mathbb{R}^{N+1}) \mid h(s) \in V(s, \mathbf{x}(s)) \text{ a.e. } s \in [0, t_0 + \delta]\}.$$

It is easy to see that $\mathcal{V}$ is closed in $L^2([0, t_0 + \delta], \mathbb{R}^{N+1})$ and, thanks to assumption (iii), $\mathcal{V}$ is also closed with respect to the weak topology. Therefore, since $\bar{h}_\varepsilon$ is equibounded in L^2, up to subsequences, $\bar{h}_\varepsilon$

converges weakly in L^2 to $\bar{h} \in \mathcal{V}$. By definition of $\mathcal{V}$, this implies that for almost every s there exist $w(s) \in \mathbf{U}$ and $\gamma(s) \geqslant 0$ such that $\bar{h}(s) = (f(\mathbf{x}(s), w(s)), L(s, \mathbf{x}(s), w(s)) + \gamma(s))$.

We claim that $\bar{h}(s) = g(s)$ for almost every $s \in [0, t_0 + \delta]$. Indeed, for almost every s, since $\bar{x}_\varepsilon$ converges uniformly to $\bar{x}$

$$|\bar{h}_\varepsilon(s) - \bar{f}(s, x_\varepsilon(s), u_\varepsilon(s))| \leqslant const \sup_{s\in[0,t_0+\delta]} |\mathbf{x}(s) - x_\varepsilon(s)| \xrightarrow[\varepsilon\to 0]{} 0.$$

Hence, by the dominated convergence theorem,

$$\int_0^t |\bar{h}_\varepsilon(s) - \bar{f}(s, x_\varepsilon(s), u_\varepsilon(s))|\, ds \xrightarrow[\varepsilon\to 0]{} 0, \quad \forall\, t \in [0, t_0 + \delta]. \tag{18}$$

For all $\varphi \in L^2([0, t_0 + \delta], \mathbb{R}^{N+1})$ and for all $t \in [0, t_0 + \delta]$ we have

$$\int_0^t \varphi(s)\bar{h}_\varepsilon(s)\, ds = \int_0^t \varphi(s)\bar{f}(s, x_\varepsilon(s), u_\varepsilon(s))\, ds + \int_0^t \varphi(s)(\bar{h}_\varepsilon(s) - \bar{f}(s, x_\varepsilon(s), u_\varepsilon(s)))\, ds \tag{19}$$

The left hand side of (19) converges to $\int_0^t \varphi(s)\bar{h}(s)\, ds$ by the weak convergence of $\bar{h}_\varepsilon$ in L^2. The right hand side of (19) converges to $\int_0^t \varphi(s)g(s)\, ds$ by the weak* convergence of $s \mapsto \bar{f}(s, x_\varepsilon(s), u_\varepsilon(s))$ in L^∞ and by (18). Hence,

$$\int_0^t \varphi(s)\bar{h}(s)\, ds = \int_0^t \varphi(s)g(s)\, ds, \quad \forall \varphi \in L^2,$$

which gives $\bar{h} = g$ almost everywhere. The existence of measurable choices of $s \mapsto w(s)$, $s \mapsto \gamma(s)$ follows from Filippov Theorem see [6, Theorem 3.1.1] and, since $\bar{h} = g$, we deduce that the trajectory corresponding to w is $\mathbf{x}$. Hence in the following we denote $\mathbf{x}$ by x_w and t_0 by t_w.

For every admissible control $v \in \mathcal{U}$ satisfying $TV(v) < \infty$ and the constraints (6),

$$\begin{aligned} \int_0^{t_w} L(t, x_w(t), w(t))\, dt &\leqslant \int_0^{t_w} (L(t, x_w(t), w(t)) + \gamma(t))dt \\ &\leqslant \limsup_{\varepsilon\to 0} \left(\int_0^{t_\varepsilon} L(t, x_\varepsilon(t), u_\varepsilon(t)) + \varepsilon TV(u_\varepsilon) \right) \\ &\leqslant \int_0^{t_v} L(t, x_v, v)\, dt, \end{aligned}$$

where the last inequality follows by (17). By density of BV in L^1 we conclude that for every control $v \in \mathcal{U}$ steering x_0 to 0 we have

$$\int_0^{t_w} L(t, x_w(t), w(t))\, dt \leqslant \int_0^{t_v} L(t, x_v, v)\, dt,$$

i.e., w is optimal for (4), (5), (6) with $\varepsilon = 0$. By uniqueness of u^* we have $w = u^*$ and $t_w = T^*$. To prove (7), it is sufficient to show that

$$\int_0^{T^*} \gamma(t)dt = 0.$$

The chain of inequalities above allows to show that for every admissible control $v \in \mathcal{U}$ such that $TV(v) < \infty$, we have

$$\int_0^{T^*} L(t, x^*(t), u^*(t))\, dt \leqslant \int_0^{t_v} L(t, x_v, v)\, dt - ||\gamma||_{L^1}. \tag{20}$$

For η small enough, let v_η be the control built in Lemma 2. If $TV(u^*) < \infty$ or u^* is chattering, then $TV(v_\eta) < \infty$. Hence applying (20) to $v = v_\eta$, we obtain that

$$\int_0^{T^*} L(t, x^*(t), u^*(t))\, dt \leqslant \int_0^{T_\eta} L(t, x_\eta, v_\eta)\, dt - ||\gamma||_{L^1}.$$

Finally, thanks to (14), when η tend to 0 we deduce $||\gamma||_{L^1} = 0$. □

4.2 Proof of Theorem 2

Comparing the statements of Theorems 1 and 2 one notices that in the latter there is no convexity assumption for velocity sets and there is no equiboundedness of trajectories required. This is due to the fact that the suboptimal controls are found by different methods: for Theorem 1 as the solution of a relaxed problem, for Theorem 2 as a truncation of the given control u^*. Nevertheless, as it is evident from the previous section (see Lemma 2), convergence of the cost along truncations of u^* was also a fundamental step in the proof of the first result. However, thanks to the regularity of the time-optimal map, in Theorem 2 not only quasi-optimality of the truncations is proved but also a convergence rate is provided.

The proof of Theorem 2 mainly follows the outline of the proof of Lemma 2. The main difference is that in this case the extra assumption that the time-optimal map is Hölder implies an explicit estimate on the rate of convergence of the costs.

Proof of Theorem 2. By assumption, there exist a ball $B_R(0)$ such that $\Upsilon \in \mathcal{C}^{0,\alpha}(B_R(0))$. Let η_0 be such that $|x^*(s)| \leqslant R$ for every $s \geqslant T^* - \eta_0$. For every $\eta \leqslant \eta_0$, apply Lemma 1 with $T_0 = 2\Upsilon(x^*(T^* - \eta))$ to the point $y = x^*(T^* - \eta)$. Denote by $w_\eta : [0, \tau_\eta] \to \mathbf{U}$ the piecewise constant control provided by Lemma 1. Then

$$\begin{aligned} \tau_\eta &\leqslant 2\Upsilon(x^*(T^* - \eta)) \leqslant 2C_\Upsilon |x^*(T^* - \eta)|^\alpha \leqslant \text{const}\, \eta^\alpha. \\ TV(w_\eta) &\leqslant M. \end{aligned}$$

Define v_η by

$$v_\eta(t) = \begin{cases} u^*(t) & \text{for } t \in [0, T^* - \eta], \\ w_\eta(t - t_\eta) & \text{for } t \in (T^* - \eta, T_\eta], \\ 0 & t \geqslant T_\eta, \end{cases} \tag{21}$$

where $T_\eta = T^* - \eta + \tau_\eta$.

Using the same argument as in the proof of Lemma 2, we obtain all the required convergences. Finally, thanks to (16), since $\tau_\eta \leqslant \text{const}\, \eta^\alpha$, (8) is proved.

□

4.3 Proof of Corollary 1

Proof of Corollary 1. For every $\eta > 0$ consider the control $w_\eta : [0, \tau_\eta] \to \mathbf{U}$, whose existence is stated in Lemma 1, steering $x^*(T^* - \eta)$ to 0 in time τ_η such that $TV(w_\eta) \leqslant \tilde{M}$. Define v_η by

$$v_\eta(t) = \begin{cases} u^*(t) & \text{for } t \in [0, T^* - \eta] \\ w_\eta(t - (T^* - \eta)) & \text{for } t \in (T^* - \eta, T^* - \eta + \tau_\eta]. \end{cases}$$

Let $T_\eta = T^* - \eta + \tau_\eta$. Note that, by construction

$$TV(v_\eta) \leqslant TV(u^*|_{[0, T^* - \eta]}) + \tilde{M}.$$

For $\varepsilon > 0$, by Theorem 1, there exists a control u_ε solution of (4), (5), (6). Therefore

$$\int_0^{T_\varepsilon} L(t, x_\varepsilon(t), u_\varepsilon(t))\, dt + \varepsilon TV(u_\varepsilon) \leqslant \int_0^{T_\eta} L(t, x_\eta(t), v_\eta(t))\, dt + \varepsilon TV(v_\eta),$$

and by Theorem 2

$$\int_0^{T_\eta} L(t, x_\eta(t), v_\eta(t))\, dt + \varepsilon TV(v_\eta) \leqslant \int_0^{T^*} L(t, x^*(t), u^*(t))\, dt + M\eta^\alpha + \varepsilon TV(v_\eta).$$

Therefore

$$\int_0^{T_\varepsilon} L(t, x_\varepsilon(t), u_\varepsilon(t))\, dt - \int_0^{T^*} L(t, x^*(t), u^*(t))\, dt \leqslant M\eta^\alpha + \varepsilon \left(TV(v_\eta) - TV(u_\varepsilon)\right). \tag{22}$$

If $TV(u^*)$ is bounded then $TV(v_\eta) \leqslant TV(u^*) + \tilde{M}$, whence $\varepsilon(TV(v_\eta) - TV(u_\varepsilon) \leqslant \varepsilon(TV(u^*) + \tilde{M})$ and we are done.

If $TV(u^*) = \infty$ then $TV(u_\varepsilon) \to \infty$ as $\varepsilon \to 0$. Indeed, if $TV(u_\varepsilon) \leqslant C$ for every $\varepsilon > 0$ then there exists a BV function $\bar{u}$ such that This implies that

$$\lim_{\varepsilon \to 0^+} \int_0^{T_\varepsilon} L(t, x_\varepsilon, u_\varepsilon) dt = \int_0^{T_{\bar{u}}} L(t, x_{\bar{u}}, \bar{u}) dt.$$

Hence $\bar{u}$ is optimal for the problem with $\varepsilon = 0$ and by uniqueness, $\bar{u} = u^*$. On the other hand, by lower semicontinuity of $TV(\cdot)$, we have that

$$TV(u^*) \leqslant \liminf_{\varepsilon \to 0} TV(u_\varepsilon) \leqslant C,$$

which is a contradiction.

Notice that the function $\eta \mapsto TV(u^*|_{[0, T^* - \eta]})$ is non-decreasing. Hence for every $\varepsilon > 0$ let $\eta(\varepsilon)$ be maximal such that $TV(u^*|_{[0, T^* - \eta(\varepsilon)]}) \leqslant TV(u_\varepsilon)$.

With this choice of η we have

$$TV(v_{\eta(\varepsilon)}) - TV(u_\varepsilon) = TV(u^*|_{[0, T^* - \eta(\varepsilon)]}) + TV(w_{\eta(\varepsilon)}) - TV(u_\varepsilon) = TV(w_{\eta(\varepsilon)}) \leqslant \tilde{M}.$$

Thus (22) with $\mu = \eta(\varepsilon)$ and taking the maximum between M and $\tilde{M}$ we have (9). □

5 Appendix: an existence result

Theorem 4. *Consider the optimal control problem*

$$\dot{x} = f(t, x, u), \quad x(0) \in M_0, \quad x(t_u) \in M_1 \tag{23}$$

$$h_1(x(t)) \geqslant 0, \ldots, h_l(x(t)) \geqslant 0 \quad \forall\, t \tag{24}$$

$$u \in \mathcal{U} = \{u : [0, t_u] \to \mathbf{U} \text{ measurable}, t_u \geqslant 0\} \tag{25}$$

$$\min_{u \in \mathcal{U}} \left(\int_0^{t_u} L(s, x(s), u(s))\, ds + \alpha TV(u) \right), \tag{26}$$

where $f : \mathbb{R} \times \mathbb{R}^N \times \mathbf{U} \to \mathbb{R}^N$ is measurable w.r.t. t, locally Lipschitz w.r.t. x, $h_1, \ldots h_l \in \mathcal{C}^0(\mathbb{R}^N)$, $L \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^m)$, $\mathbf{U} \subset \mathbb{R}^m$ is compact, M_0, M_1 are compact submanifolds, and $\alpha > 0$. Assume that *(i)* $\forall\, x \in M^0 \cup M^1$, $h_1(x) \geqslant 0, \dots h_l(x) \geqslant 0$;

(ii) *there exists* $\bar{u} \in \mathcal{U}$ *having bounded variation, steering* M_0 *to* M_1*, and such that the corresponding trajectory satisfies the constraints* (24)*;*

(iii) *there exists* $b > 0$ *such that, for every* $u \in \mathcal{U}$ *and every trajectory* x_u *of* (23) *corresponding to* u*, we have*

$$t_u + ||x_u||_\infty \leqslant b.$$

Then the optimal control problem (23), (24), (26) *admits a solution.*

Proof. Let

$$\delta = \inf \left(\int_0^{t_u} L(s, x(s), u(s))\, ds + \alpha TV(u) \right),$$

where the infimum is taken among all controls $u \in \mathcal{U}$ steering M_0 to M_1 and whose corresponding trajectory satisfies (24). Let $u_n : [0, T_n] \to \mathbf{U}$ be a minimizing sequence of admissible controls, i.e.,

$$\lim_{n\to\infty} \left(\int_0^{T_n} L(s, x_n(s), u_n(s))\, ds + \alpha TV(u_n) \right) = \delta,$$

where x_n is a trajectory corresponding to u_n. Then, thanks to assumption (iii), for n sufficiently large we have that

$$\alpha TV(u_n) \leqslant \int_0^{t_{\bar{u}}} L(s, x_{\bar{u}}(s), \bar{u}(s))\, ds + \alpha TV(\bar{u}) + C,$$

for some constant $C \geqslant 0$, which implies that u_n is a bounded sequence in BV. Then by a standard compactness result in BV there exists a subsequence, still denoted by u_n, and a control $u_\alpha \in \mathcal{U}$ such that

$$\lim_{n\to\infty} ||u_n - u_\alpha||_{L^1} = 0.$$

Up to a subsequence, let u_n converge to u_α almost everywhere, T_n converge to t_α and $x_n(0)$ converge to x_α^0. We are going to prove that $u_\alpha : [0, t_\alpha] \to \mathbf{U}$ is a solution to (23), (24), (26). By [17, Theorem 1 p. 56], the convergence almost everywhere of u_n to u_α implies that if x_α satisfies

$$\dot{x}_\alpha = f(t, x_\alpha, u_\alpha), \quad x_\alpha(0) = x_\alpha^0$$

then x_n converges uniformly to x_α. Hence x_α satisfies the constraints (24) and, by compactness of M_0 and M_1, we obtain that $x_\alpha(t_\alpha) \in M_1$. Hence u_α is admissible. Moreover,

$$\lim_{n\to\infty} L(t, x_n(t), u_n(t)) = L(t, x_\alpha(t), u_\alpha(t)), \qquad \text{for almost every } t.$$

Hence, thanks to assumption (iii), the dominated convergence theorem allows to conclude that

$$\lim_{n\to\infty} \int_0^{T_n} L(t, x_n(t), u_n(t))\, dt = \int_0^{t_\alpha} L(t, x_\alpha(t), u_\alpha(t))\, dt \tag{27}$$

On the other hand, by lower semicontinuity of $TV(\cdot)$ we have

$$TV(u_\alpha) \leqslant \liminf_{n\to\infty} TV(u_n). \tag{28}$$

Using (27), (28) and since u_n is a minimizing sequence, we infer that

$$\int_0^{t_\alpha} L(t, x_\alpha(t), u_\alpha(t))\, dt + \alpha TV(u_\alpha) \leqslant \delta$$

which implies that u_α is optimal. □

In contrast with usual existence results see [7], we do not assume convexity of the dynamics. The conclusion is assured by the term of the total variation in the cost functional.

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