A Normal Form for Generic 2-Dimensional Almost-Riemannian Structures at a tangency point

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Abstract

Two-dimensional almost-Riemannian structures are generalized Riemannian structures on surfaces for which a local orthonormal frame is given by a Lie bracket generating pair of vector fields that can become collinear. Generically, there are three type of points: Riemannian points where the two vector fields are linearly independent, Grushin points where the two vector fields are collinear but their Lie bracket is not and tangency points where the two vector fields and their Lie bracket span a one-dimensional space and the missing direction is obtained with one more bracket.

In this paper we consider the problem of finding a normal form at a generic tangency point. The problem happens to be equivalent to finding a smooth canonical parameterized curve passing through the point and transversal to the distribution. It is known that the cut locus from the point is not a good candidate since it is not smooth. Therefore, we analyse the cut locus from the singular set and we prove that it is not smooth either. A good candidate appears to be a crest of the Gaussian curvature. Such crest is uniquely determined and has a natural parametrization.

1 Introduction

A 2-dimensional Almost Riemannian Structure (2-ARS for short) is a rank-varying sub-Riemannian structure that can be defined locally by a pair of smooth vector fields on a 2-dimensional manifold, satisfying the Hörmander condition. These vector fields play the role of an orthonormal frame.

Let us denote by $\Delta(q)$ the linear span of the two vector fields at a point q. Where $\Delta(q)$ is 2-dimensional, the corresponding metric is Riemannian. Where $\Delta(q)$ is 1-dimensional, the corresponding Riemannian metric is not well defined, but thanks to the Hörmander condition one can still define the Carnot-Caratheodory distance between two points, which happens to be finite and continuous.

2-ARSs were introduced in the context of hypoelliptic operators [17, 18], they appeared in problems of population transfer in quantum systems [14, 13, 12], and have applications to orbital

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transfer in space mechanics [9, 8]. 2-ARSs are a particular case of Rank-varying sub-Riemannian structures (see for instance [7, 19, 23]).

Generically, the singular set $\mathcal{Z}$, where $\Delta(q)$ has dimension 1, is a 1-dimensional embedded submanifold (see [3]) and there are three types of points: Riemannian points, Grushin points where $\Delta(q)$ is 1-dimensional and transversal to $\mathcal{Z}$, and tangency points where $\Delta(q)$ is tangent to $\mathcal{Z}$, these last ones being isolated.

2-ARSs present very interesting phenomena. For instance, the presence of a singular set permits the conjugate locus to be nonempty even if the Gaussian curvature is negative, where it is defined (see [3]). Moreover, a Gauss–Bonnet-type formula can be obtained. More precisely, in [3, 16] the authors studied the generic case without tangency points. In [5] this formula was generalized to the case in which tangency points are present. (For generalizations of Gauss–Bonnet formula in related contexts, see also [4, 20, 21].) In [15] a necessary and sufficient condition for two 2-ARSs on the same compact manifold M to be Lipschitz equivalent was given. This equivalence was established in terms of graphs associated with the structures.

Tangency points are the most difficult to handle due to the fact that the asymptotic of the distance is different from the two sides of the singular set. In [10] the authors gave a description of the geometry of the nilpotent approximation at a tangency point, provided jets of the exponential map and a description of the cut and conjugate loci from a tangency point in the generic case.

However, tangency points are far to be deeply understood. An open question is the convergence or the divergence of the integral of the geodesic curvature on the boundary of a tubular neighborhood of the singular set, close to a tangency point. This question arose in the proof of the Gauss–Bonnet theorem given in [5]. In that paper, thanks to numerical simulations, the authors conjecture the divergence of such integral.

Another open question is how to find a normal form for the orthonormal frame at tangency points which is completely reduced, in the sense that it depends only on the 2-ARS and not on its local representation. In [3] the local representations given in Figure 1 were found, but the ones corresponding to Riemannian and tangency points are not completely reduced. Indeed, there exist change of coordinates and rotations of the frame for which an orthonormal base has the same expression as in (F1) (resp. (F3)), but with a different function ϕ (resp. with different functions ψ and ξ).

In order to build the coordinate system for which the local expressions found in [3] apply, the idea was the following. Consider a smooth parametrized curve passing through a point q. If the curve is assumed to be transversal to the distribution at each point, then the Carnot–Caratheodory distance from the curve is shown to be smooth on a neighborhood of q (see [3]). Given a point p near q the first coordinate of p is, by definition, the distance between p and the chosen curve, with a suitable choice of sign. The second coordinate of p is the parameter corresponding to the point on the chosen curve that realizes the distance between p and the curve. If the parameterized curve used in this construction can be built canonically, then one gets a normal form that cannot be further reduced.

For Riemannian and Grushin points, a canonical parametrized curve transversal to the distribution can be easily identified, at least in the generic case (see Sections 3 and the Appendix). For Grushin points, a canonical curve transversal to the distribution is the set $\mathcal{Z}$. This curve has also a natural parameterization as explained below Proposition 2.

As concerns the local expression (F3) (see Figure 1), in [3] the choice of the smooth parametrized curve was arbitrary and not canonical. The main purpose of this paper is to find a canonical one.

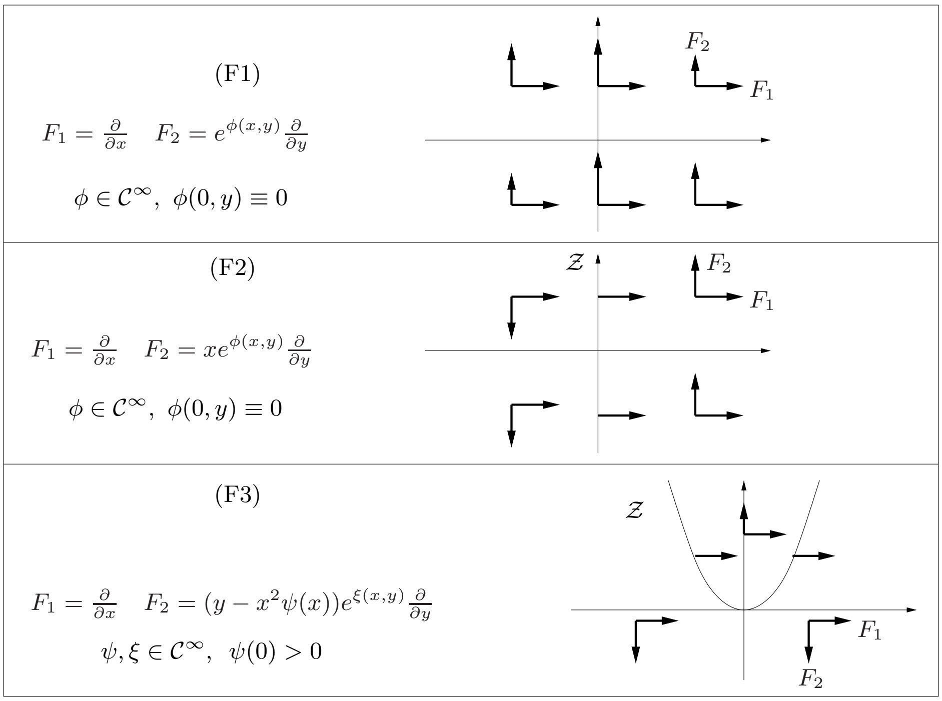

Figure 1: The local representations established in [3]

In other words, to identify the true invariants of the structure at a tangency point.

The most natural candidate for such a curve is the cut locus from the tangency point. Nevertheless, this is not a good choice, as in [10] it was proved that in general the cut locus starting from the point is not smooth but has an asymmetric cusp (see Figure 2). Another possible candidate is the cut locus from the singular set in a neighborhood of the tangency point. The first result of the paper concerns the analysis and description of this locus (see Proposition 5). The crucial fact is that we prove the cut locus from $\mathcal{Z}$ in a neighborhood of a tangency point to be non-smooth (see Figure 2).

A third possibility is to look for crests or valleys of the Gaussian curvature which intersect transversally the singular set at a tangency point. The second result of the paper (see Theorem 1) consists in the proof of the existence of such a crest. Moreover, this curve admits a canonical regular parameterization. Then, a completely reduced normal form is obtained implicitly by requiring this curve to be the vertical axis. However, explicit relations between the Taylor coefficients of the functions ψ and ξ at the point can be obtained.

The structure of the paper is the following. In section 2 we briefly recall the notion of almost-Riemannian structure. Section 3 is devoted to the analysis of local representations of a 2-ARS at a point and contains the statements of the main results. The proposition describing the cut locus from the singular set in a neighborhood of a tangency point is proven in Section 4. The main result, which provides an intrinsic parameterized curve transversal to the distribution at a tangency point and a relation between the functions ψ and ξ, is proven in Section 5. Finally, in Section 6 we study the case of Riemannian points. For points such that the gradient of the curvature is nonzero, we provide a completely reduced normal form. For generic critical points of the curvature, we find a relation for the function ϕ.

2 Preliminaries

In this section we recall some basic definitions in the framework of 2-ARS following [5, 3].

Let M be a smooth surface without boundary. Throughout the paper, unless specified, manifolds are smooth (i.e., $\mathcal{C}^\infty$) and without boundary; vector fields and differential forms are smooth. The set of smooth vector fields on M is denoted by $\mathrm{Vec}(M)$.

Definition 1 *A 2-dimensional almost-Riemannian structure (2-ARS) is a triple $\mathcal{S} = (E, \mathfrak{f}, \langle\cdot,\cdot\rangle)$ where E is a vector bundle of rank 2 over M and $\langle\cdot,\cdot\rangle$ is a Euclidean structure on E, that is, $\langle\cdot,\cdot\rangle_q$ is a scalar product on E_q smoothly depending on q. Finally $\mathfrak{f} : E \to TM$ is a morphism of vector bundles, i.e.,* **(i)** *the diagram*

$$\begin{tikzcd} E \arrow[r, "\mathfrak{f}"] \arrow[dr, "\pi_E"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes, where $\pi : TM \to M$ and $\pi_E : E \to M$ denote the canonical projections and **(ii)** *$\mathfrak{f}$ is linear on fibers. Denoting by $\Gamma(E)$ the $\mathcal{C}^\infty(M)$-module of smooth sections on E, we require the morphism $\mathfrak{f}_* : \Gamma(E) \to \mathrm{Vec}(M)$, $\mathfrak{f}_*(\sigma) = \mathfrak{f} \circ \sigma$ to be injective. Moreover, we assume the submodule $\Delta = \mathfrak{f}_*(\Gamma(E))$ to be bracket generating, i.e., $Lie_q(\Delta) = T_qM$ for every $q \in M$.*

A property (P) defined for 2-ARSs is said to be *generic* if for every rank-2 vector bundle E over M, (P) holds for every $\mathfrak{f}$ in an open and dense subset of the set of morphisms of vector bundles from E to TM, endowed with the $\mathcal{C}^\infty$-Whitney topology.

Let $\mathcal{S} = (E, \mathfrak{f}, \langle\cdot,\cdot\rangle)$ be a 2-ARS on a surface M. We denote by $\Delta(q)$ the linear subspace $\{V(q) \mid V \in \Delta\} = \mathfrak{f}(E_q) \subseteq T_qM$. The set of points in M such that $\dim(\Delta(q)) < 2$ is called *singular set* and denoted by $\mathcal{Z}$. The Euclidean structure on E induces a symmetric positive-definite bilinear form $G : \Delta \times \Delta \to \mathcal{C}^\infty(M)$ defined by $G(V, W) = \langle\sigma_V, \sigma_W\rangle$ where σ_V, σ_W are the unique sections of E satisfying $\mathfrak{f} \circ \sigma_V = V, \mathfrak{f} \circ \sigma_W = W$. At points $q \in M$ where $\mathfrak{f}|_{E_q}$ is an isomorphism, G is a tensor and the value $G(V, W)|_q$ depends only on $V(q), W(q)$. This is no longer true at points q where $\mathfrak{f}|_{E_q}$ is not injective.

If (σ_1, σ_2) is an orthonormal frame for $\langle\cdot,\cdot\rangle$ on an open subset Ω of M, an *orthonormal frame for G* on Ω is given by $(\mathfrak{f} \circ \sigma_1, \mathfrak{f} \circ \sigma_2)$.

For every $q \in M$ and every $v \in \Delta(q)$ define $\mathbf{G}_q(v) = \inf\{\langle u, u\rangle_q \mid u \in E_q, \mathfrak{f}(u) = v\}$.

An absolutely continuous curve $\gamma : [0, T] \to M$ is admissible for $\mathcal{S}$ if there exists a measurable essentially bounded function $[0, T] \ni t \mapsto u(t) \in E_{\gamma(t)}$ such that $\dot\gamma(t) = \mathfrak{f}(u(t))$ for almost every $t \in [0, T]$. Given an admissible curve $\gamma : [0, T] \to M$, the *length of* γ is

$$\ell(\gamma) = \int_0^T \sqrt{\mathbf{G}_{\gamma(t)}(\dot\gamma(t))}\, dt.$$

The *Carnot-Caratheodory distance* (or sub-Riemannian distance) on M associated with $\mathcal{S}$ is defined as

$$d(q_0, q_1) = \inf\{\ell(\gamma) \mid \gamma(0) = q_0, \gamma(T) = q_1, \gamma \text{ admissible}\}.$$

The finiteness and the continuity of $d(\cdot,\cdot)$ with respect to the topology of M are guaranteed by the Lie bracket generating assumption on the rank-varying sub-Riemannian structure (see [6]). The Carnot-Caratheodory distance endows M with the structure of metric space compatible with the topology of M as differential manifold.

A *geodesic* for $\mathcal{S}$ is an admissible curve $\gamma:[0,T]\to M$, such that $\mathbf{G}_{\gamma(t)}(\dot\gamma(t))$ is constant and for every sufficiently small interval $[t_1,t_2]\subset[0,T]$ $\gamma|_{[t_1,t_2]}$ is a minimizer of ℓ. A geodesic for which $\mathbf{G}_{\gamma(t)}(\dot\gamma(t))$ is (constantly) equal to one is said to be parameterized by arclength.

Locally, in an open set Ω, if (F_1,F_2) is an orthonormal frame, a curve parameterized by arclength is a geodesic if and only if it is the projection on Ω of a solution of the Hamiltonian system corresponding to the Hamiltonian

$$H(q,\mathbf{p})=\frac{1}{2}((\mathbf{p}F_1(q))^2+(\mathbf{p}F_2(q))^2),\quad q\in\Omega,\mathbf{p}\in T_q^*\Omega.$$

lying on the level set $H=1/2$. This is the Pontryagin Maximum Principle [22] in the case of 2-ARS. Its simple form follows from the absence of abnormal extremals in 2-ARS, as a consequence of the Hörmander condition see [3]. When looking for geodesics γ minimizing the distance from a submanifold (possibly of dimension zero) $\mathcal{T}$, one should add the transversality condition $\mathbf{p}(0)T_{\gamma(0)}\mathcal{T}=0$.

The cut locus $\mathbf{K}_\mathcal{T}$ from $\mathcal{T}$ is the set of points p for which exists a geodesic realizing the distance between $\mathcal{T}$ and p which loses optimality after p.

It is well known (see for instance [1] for a proof in the three-dimensional contact case) that if $p\in\mathbf{K}_\mathcal{T}$ then one of the following two possibilities happen: **i)** more than one minimizing geodesics reach p; **ii)** p belongs to the first conjugate locus from $\mathcal{T}$ defined as follows. To simplify the notation, assume that all geodesics are defined in $[0,\infty[$. Define

$$C^0=\{(q,\mathbf{p})\in T^*\Omega\mid q\in\mathcal{T},\ H(q,\mathbf{p})=1/2,\ \mathbf{p}T_q\mathcal{T}=0\}$$

and

$$\begin{aligned}\exp: C^0\times[0,\infty[&\to M\\ \lambda&\mapsto\pi(e^{t\vec{H}}\lambda)\end{aligned}$$

where π is the canonical projection $(q,\mathbf{p})\to q$ and $\vec{H}$ is the Hamiltonian vector field corresponding to H. The first conjugate time is

$$t(\lambda)=\min\{t>0,(\lambda,t)\text{ is a critical point of }\exp\}.$$

and the first conjugate locus is $\{\exp(\lambda,t(\lambda))\mid\lambda\in C^0\}$.

3 Normal forms and main results

Let us introduce the main assumptions under which all the results of the paper are proven.

We say that a 2-ARS *satisfies condition* **(H0)** if the following properties hold: **(i)** $\mathcal{Z}$ is an embedded one-dimensional submanifold of M; **(ii)** the points $q\in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; **(iii)** $\Delta_3(q)=T_qM$ for every $q\in M$, where $\Delta_1=\Delta$ and $\Delta_{k+1}=\Delta_k+[\Delta,\Delta_k]$. The property **(H0)** is generic for 2-ARSs.

3.1 Completely reduced normal forms

Definition 2 *A local representation of a 2-ARS at a point $q \in M$ is a pair of vector fields (X, Y) on $\mathbf{R}^2$ such that there exist:* **i)** *a neighborhood U of q in M, a neighborhood V of $(0,0)$ in $\mathbf{R}^2$ and a diffeomorphism $\varphi : U \to V$ such that $\varphi(q) = (0,0)$;* **ii)** *a local orthonormal frame (F_1, F_2) of Δ around q, such that $\varphi_* F_1 = X$, $\varphi_* F_2 = Y$, where φ_* denotes the push-forward.*

Definition 3 *We say that a local representation is completely reduced if it corresponds to a canonical choice of φ, F_1 and F_2, up to orientation. In this case we call the pair (X, Y) a completely reduced normal form.*

Remark 1 *The expression "Up to orientation" is necessary since, if $\varphi = (\varphi_1, \varphi_2)$, there is no way in general to choose canonically among (φ_1, φ_2), $(-\varphi_1, \varphi_2)$, $(-\varphi_1, -\varphi_2)$, $(\varphi_1, -\varphi_2)$, and among (F_1, F_2), $(-F_1, F_2)$, $(-F_1, -F_2)$, $(F_1, -F_2)$.*

Remark 2 *Here by canonical we mean that it depends only on the 2-ARS. In this case if $X = (X_1(x,y), X_2(x,y))$ and $Y = (Y_1(x,y), Y_2(x,y))$ then up to sign and up to the transformations $x \to -x$ and $y \to -y$, we have that $X_1(x,y)$, $X_2(x,y)$, $Y_1(x,y)$, $Y_2(x,y)$ are functional invariants of the system.*

Proposition 1 *Under the hypothesis* **(H0)**, *it is always possible to get a local representation in the form $X = \partial_x$, $Y = f(x,y)\partial_y$, where f is a smooth function such that one of the following conditions holds: $f(0,0) \neq 0$, $\partial_x f(0,0) \neq 0$, $\partial_{xx} f(0,0) \neq 0$.*

The proof of this proposition can be found in [3]. The first step of the proof consists in choosing a parameterized curve C transversal to Δ at q and in defining locally around q a function sign whose value is $+1$ on one side and -1 on the other. The second step consists in proving that the function δ obtained by multiplying sign by the distance function to the curve C is smooth around q. The third step consists in choosing as first coordinate of a point p the value of $\delta(p)$ and as second coordinate the parameter of the point on the curve C which realizes such distance. In this system of coordinates an orthonormal frame has the form given in the proposition above.

Conversely, for a local representation of the form $X = \partial_x$, $Y = f(x,y)\partial_y$, the vertical axis is transversal to the distribution and $|x|$ is the distance of a point (x,y) from it.

Hence we have the following:

Claim: up to orientation, constructing a local representation of the form $X = \partial_x$, $Y = f(x,y)\partial_y$ is equivalent to choose a parameterized curve transversal to the distribution.

Thanks to the claim, constructing a completely reduced normal form of the type $(\partial_x, f(x,y)\partial_y)$ is equivalent to choose a canonical parameterized curve transversal to the distribution.

In [3] the following local representations were constructed.

Proposition 2 *If a 2-ARS satisfies* **(H0)**, *then for every point $q \in M$ there exist a neighborhood U of q and an orthonormal frame (F_1, F_2) of the ARS on U such that, up to a change of coordinates, $q = (0,0)$ and (F_1, F_2) has one of the forms*

$$\begin{array}{lll} (\mathbf{F}1) & F_1(x,y) = \frac{\partial}{\partial x}, & F_2(x,y) = e^{\phi(x,y)}\frac{\partial}{\partial y}, \\ (\mathbf{F}2) & F_1(x,y) = \frac{\partial}{\partial x}, & F_2(x,y) = x e^{\phi(x,y)}\frac{\partial}{\partial y}, \\ (\mathbf{F}3) & F_1(x,y) = \frac{\partial}{\partial x}, & F_2(x,y) = (y - x^2\psi(x)) e^{\xi(x,y)}\frac{\partial}{\partial y}, \end{array}$$

where ϕ, ψ and ξ are smooth functions such that $\phi(0,y)=0$ and $\psi(0)>0$.

A point q is said to be *Riemannian* if $\Delta(q)=T_qM$, *Grushin point* if $\Delta(q)$ is one-dimensional and $\Delta_2(q)=T_qM$, *tangency* point if $\Delta(q)=\Delta_2(q)$ and $\Delta_3(q)=T_qM$. If the structure satisfies **(H0)**, then a local representation for a Riemannian, Grushin, tangency point is (F1), (F2), (F3) respectively.

In the local representations given in Proposition 2, (F2) is completely reduced. Indeed, in the proof of Proposition 2 (see [3]) the authors chose as curve C the singular set $\mathcal{Z}$, which is naturally associated to the structure. It is easy to see that for any orthonormal frame (G_1,G_2), the Lie bracket $[G_1,G_2]|_{\mathcal{Z}}$ modulo elements in Δ does not change. As for the parametrization of $\mathcal{Z}$, the choice in [3] was such that $[F_1,F_2]|_{\mathcal{Z}}=\frac{\partial}{\partial y}$ modulo Δ. For what concerns (F1) and (F3), they are not completely reduced since the curve transversal to the distribution is arbitrary. Our aim is to provide at Riemannian and tangency points a canonical choice of a parametrized curve associated with the structure.

First, let us consider the case of Riemannian points. Generically, the set of Riemannian points $p\in M$ such that the gradient of the Gaussian curvature K is singular is a discrete set Π, and at each point of this discrete set, exactly one crest and one valley of K passes through the point. Hence, at a point outside Π, one can choose as C the level set of the curvature, parameterized by arclength. For points of Π, one can choose the crest or the valley parameterized by arclength.

In the following proposition we sum up the analysis of normal forms at Riemannian points. For the sake of readability, the proof is postponed to Section 6.

Proposition 3 *Let $q\in M$ be a Riemannian point of a generic 2-ARS. If $\nabla K(q)\neq 0$, then a completely reduced normal form for $\mathcal{S}$ at q is (F1) where $\phi(0,y)\equiv 0$ and*

$$-2\partial_x^2\phi(0,y)\partial_x\partial_y\phi(0,y)+\partial_x^2\partial_y\phi(0,y)\equiv 0.$$

If $\nabla K(q)=0$, then a local representation for $\mathcal{S}$ at q is (F1) where $\phi(0,y)\equiv 0$ and $h_0=0$ where h_0 is defined in formula (12).

The case of tangency points is rather complicated. The first candidate as smooth curve is the cut locus from the tangency point. Let us recall a result of [10] where the shape of the cut locus at a tangency point has been computed.

Proposition 4 *Let $\mathcal{S}$ be a 2-ARS on M satisfying* **(H0)**. *Let $q\in M$ be a tangency point such that there exists a local representation of the type (F3) for $\mathcal{S}$ at q with the property*

$$\psi'(0)+\psi(0)\partial_x\xi(0,0)\neq 0.$$

Then the cut locus from the tangency point accumulates at q as an asymmetric cusp whose branches are separated locally by $\mathcal{Z}$. In the coordinates system where the chosen local representation is (F3), the cut locus accumulates as the set

$$\{(x,y)\mid y>0, y^2-\alpha_1x^3=0\}\cup\{(x,y)\mid y<0, y^2-\alpha_2x^3=0\},$$

with $\alpha_i=c_i/(\psi'(0)+\psi(0)\partial_x\xi(0,0))^3$, the constants c_i being nonzero and independent on the structure.

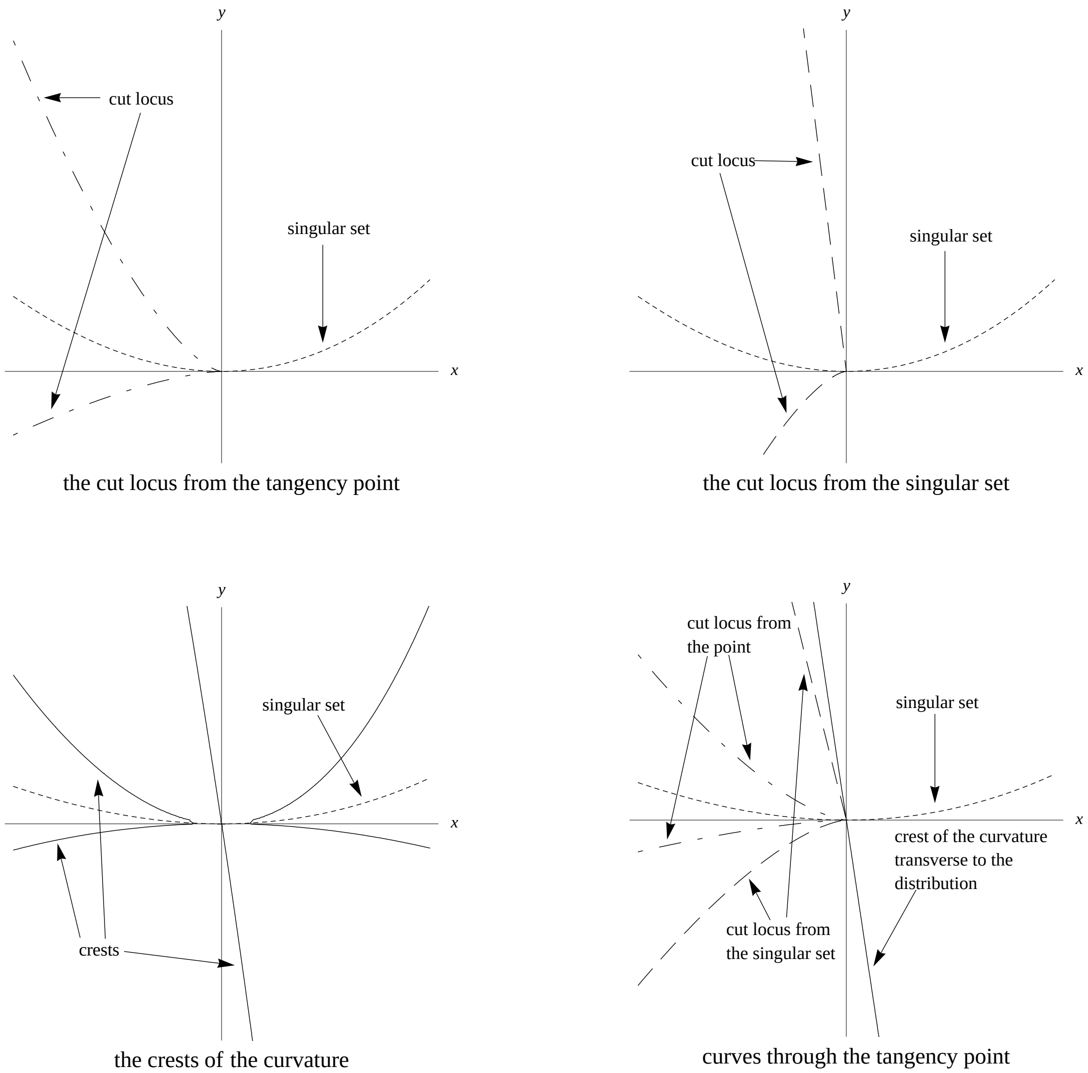

Figure 2: The singular locus (dotted line), the cut locus from a tangency point (semidashed line), the cut locus from the singular set (dashed line), and the crests of the Gaussian curvature (solid lines) for the ARS with orthonormal frame $F_1 = \frac{\partial}{\partial x}$, $F_2 = (y - x^2 - x^3)\frac{\partial}{\partial y}$. In this case there are three crests of the curvature. Notice that all the crests except only one are tangent to the distribution.

Due to Proposition 4, in general the cut locus is not smooth and cannot be used to find a completely reduced normal form. Another candidate would be the cut locus from $\mathcal{Z}$ in a neighborhood of a tangency point. A description of such locus is given by the following proposition.

Proposition 5 *Let $q \in M$ be a tangency point of a 2-ARS satisfying hypothesis* **(H0)** *and assume there exists a local representation of the type (F3) for $\mathcal{S}$ at q with the property*

$$\psi'(0) + \psi(0)\partial_x \xi(0,0) \neq 0.$$

Then the cut locus from the singular set $\mathcal{Z}$ in a neighborhood of q accumulates at q as the union of two curves locally separated by $\mathcal{Z}$, one converging to q transversally to $\mathcal{Z}$, the other one with tangent direction at q belonging to the distribution. In the chosen local representation, the tangent line at q to the part of the cut locus which is transverse to the distribution is $x = -\frac{1}{2}\psi'(0)y$.

The proof Proposition 5 is in the same spirit of the proof of Proposition 4 (see [10]) and it is postponed to the next section. Notice that, being non-smooth at the tangency point, the cut locus from $\mathcal{Z}$ cannot be chosen to build a completely reduced normal form.

Finally, we look for a candidate curve to build a completely reduced normal form among the crests or valleys of the curvature. Generically, a crest passing through a tangency point, being smooth and transverse to the distribution, happens to exist and to be unique. Moreover we prove that along this curve the scalar product between the tangent vector to the curve and the gradient of the curvature is smooth and nonvanishing when prolonged to the tangency point. Requiring it to be identically equal to 1, we fix a canonical parameterization. More precisely we get the following result.

Theorem 1 *There exists $\epsilon > 0$ and a unique smooth parametrized curve Γ defined on $(-\epsilon, \epsilon)$ which satisfies the following properties: (i) $\Gamma(0) = q$, $\Gamma'(0) \notin \Delta(q)$; (ii) the support of Γ is contained in a crest of the Gaussian curvature K; (iii) $G(\Gamma'(t), \nabla K(\Gamma(t))) \equiv 1$.*

Remark 3 *Notice that crests and valleys of the curvature are included in the set*

$$\{p \in M \mid G(\nabla(||\nabla K||^2), \nabla K^\perp) = 0\},$$

where ∇K denotes the almost-Riemannian gradient of K, i.e., the unique vector such that $G(\nabla K, \cdot) = dK(\cdot)$, $||\nabla K||^2 = G(\nabla K, \nabla K)$, and $(\nabla K)^\perp$ satisfies $G(\nabla K, (\nabla K)^\perp) = 0$.

This curve can be used to reduce completely the local representation (F3). Unfortunately, since in the proof of Theorem 1 the crest is obtained as an implicit solution of the equation given in **ii)**, we cannot get explicitly the relations between the functions ψ and ξ. However, one can compute relations among the Taylor coefficients of them. For instance, at the first order we get

Proposition 6 *In the local representation (F3) we can choose the functions ξ, ψ such that $2\xi_x(0,0)\psi(0) - 3\psi'(0) = 0$.*

The proof of Theorem 1 is given in Section 5.

4 The cut locus from the singular set

In this section we prove Proposition 5 starting from the local representation (F3). Notice that applying the coordinate change

$$\tilde{x} = x, \quad \tilde{y} = \frac{y}{\psi(0)},$$

we may assume $\psi(0) = 1$. For the sake of readability, in the following we rename $\tilde{x}, \tilde{y}$ by x, y. Since $\psi(0) > 0$, the singular set $\mathcal{Z}$ is locally contained in the upper half plane $\{(x, y) \mid y \geq 0\}$.

Locally, the singular set separates M in two domains $\{(x, y) \mid y - x^2\psi(x) > 0\}$ and $\{(x, y) \mid y - x^2\psi(x) < 0\}$. Notice that since we are computing $\mathbf{K}_\mathcal{Z}$, the cut locus from $\mathcal{Z}$, we have $\mathbf{K}_\mathcal{Z} \cap \mathcal{Z} = \emptyset$.

Moreover the only point of $\mathcal{Z}$ where $\mathbf{K}_{\mathcal{Z}}$ can accumulate is the tangency point, since all other points of $\mathcal{Z}$ are Grushin points, where Δ is transversal to $\mathcal{Z}$. Hence, locally, $\mathbf{K}_{\mathcal{Z}}$ is the union of two parts, $\mathbf{K}_{\mathcal{Z}}^{+}$ lying in the upper domain and $\mathbf{K}_{\mathcal{Z}}^{-}$ in the lower one.

As we shall see, the two components of $\mathbf{K}_{\mathcal{Z}}$ have different natures: in the upper domain the geodesic starting at a point $(a, a^2\psi(a))$ and minimizing the distance from $\mathcal{Z}$ reaches its cut point at a time of order 1 in $|a|$, when in the lower domain the geodesic starting at the same point reaches its cut point at a time of order 1 in $\sqrt{|a|}$.

Applying the Pontryagin Maximum Principle, geodesics for the ARS are projections on $\mathbf{R}^2$ of solutions of the Hamiltonian system associated with the function

$$H = \frac{1}{2}(p_x^2 + p_y^2(y - x^2\psi(x))^2 \mathrm{e}^{2\xi(x,y)}),$$

that is, solutions of the system

$$\left\{\begin{array}{rcl} \dot{x} & = & p_x \\ \dot{y} & = & p_y((y - x^2\psi(x))\mathrm{e}^{\xi(x,y)})^2 \\ \dot{p}_x & = & p_y^2(y - x^2\psi(x))(2x\psi(x) + x^2\psi'(x) - (y - x^2\psi(x))\frac{\partial\xi}{\partial x}(x,y))\mathrm{e}^{2\xi(x,y)} \\ \dot{p}_y & = & -p_y^2(y - x^2\psi(x))(1 + (y - x^2\psi(x))\frac{\partial\xi}{\partial y}(x,y))\mathrm{e}^{2\xi(x,y)}. \end{array}\right. \tag{1}$$

In addition, a solution with $x(0) = a, y(0) = a^2\psi(a)$, $a \neq 0$ and minimizing the distance from $\mathcal{Z}$ must satisfy the transversality condition

$$p_x(0) = \pm 1, \quad p_y(0) = \mp\frac{1}{2a\psi(a) + a^2\psi'(a)}.$$

Introducing the new time variable $s = \frac{t}{\eta}$ where $\eta > 0$ is a parameter, system (1) becomes

$$\left\{\begin{array}{rcl} \frac{dx}{ds} & = & \eta p_x \\ \frac{dy}{ds} & = & \eta p_y((y - x^2\psi(x))\mathrm{e}^{\xi(x,y)})^2 \\ \frac{dp_x}{ds} & = & \eta p_y^2(y - x^2\psi(x))(2x\psi(x) + x^2\psi'(x) - (y - x^2\psi(x))\frac{\partial\xi}{\partial x}(x,y))\mathrm{e}^{2\xi(x,y)} \\ \frac{dp_y}{ds} & = & -\eta p_y^2(y - x^2\psi(x))(1 + (y - x^2\psi(x))\frac{\partial\xi}{\partial y}(x,y))\mathrm{e}^{2\xi(x,y)}. \end{array}\right. \tag{2}$$

The proof of the result splits in two steps, where we describe first $\mathbf{K}_{\mathcal{Z}}^{+}$ and then $\mathbf{K}_{\mathcal{Z}}^{-}$.

In each step we proceed as follows: first we compute jets of the exponential map; second we try to identify which geodesics intersect at the same time t; finally we check that the conjugate time of these geodesics is bigger than t.

4.1 The upper part of the cut locus

We consider the geodesic starting from a point of $\mathcal{Z}$ with initial condition

$$x(0) = a > 0,\ y(0) = a^2\psi(a),\ p_x(0) = -1,\ p_y(0) = \frac{1}{2a\psi(a) + a^2\psi'(a)}, \tag{3}$$

i.e., the geodesic realizing locally the distance from $\mathcal{Z}$ and entering the upper domain. Taking $\eta = a$, one can check that if x, y, p_x, p_y have orders $1, 2, 0, -1$ in η respectively, then the dynamics has the same or higher orders. As a consequence, since the initial conditions respect these orders, one can compute jets with respect to η of the solution of system (2) under the form

$$\begin{array}{rclrcl} x(s) & = & \eta x_0(s) + \eta^2 x_1(s) + \eta^3 \bar{x}(\eta, s) & p_x(s) & = & p_{x0}(s) + \eta p_{x1}(s) + \eta^2 \bar{p}_x(\eta, s) \\ y(s) & = & \eta^2 y_0(s) + \eta^3 y_1(s) + \eta^4 \bar{y}(\eta, s) & p_y(s) & = & \eta^{-1} p_{y0}(s) + p_{y1}(s) + \eta \bar{p}_y(\eta, s) \end{array}$$

where $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ are smooth functions. Using (3), the initial conditions are given by

$$\begin{array}{llll} x_0(0) = 1, & x_1(0) = 0, & p_{x0}(0) = -1, & p_{x1}(0) = 0 \\ y_0(0) = 1 & y_1(0) = \psi'(0), & p_{y0}(0) = \frac{1}{2}, & p_{y1} = -\frac{3}{4}\psi'(0), \end{array}$$

and from system (2) we easily get

$$x_0(s) = 1 - s, \quad x_1(s) \equiv 0, \quad y_0(s) \equiv 1, \quad y_1(s) \equiv \psi'(0),$$

whence

$$x(t) = a - t + a^3 \bar{x}(a, t/a), \quad y(t) = a^2 + a^3 \psi'(0) + a^4 \bar{y}(a, t/a).$$

Similarly, the solution of (1) with initial condition

$$x(0) = a < 0, \; y(0) = a^2 \psi(a), \; p_x(0) = 1, \; p_y(0) = \frac{1}{2a\psi(a) + a^2 \psi'(a)}, \tag{4}$$

satisfies

$$x(t) = a + t + a^3 \bar{x}(a, t/a), \quad y(t) = a^2 + a^3 \psi'(0) + a^4 \bar{y}(a, t/a).$$

This allows to prove that, for any $c > 0$, at t fixed, $\frac{\partial x(t)}{\partial a} > \frac{1}{2}$ for $0 < |\frac{t}{a}| < c$ and a small enough. Hence two geodesics starting with two initial conditions a and $\bar{a}$ of the same sign do not intersect at time t if a and $\bar{a}$ are small enough and $|\frac{t}{a}|$ and $|\frac{t}{\bar{a}}|$ are less than c.

For what concerns the conjugate locus, the Jacobian of the map $(a, t) \mapsto (x(t), y(t))$ is $2a + 3a^2\psi'(0) + a^3 \Xi(a, \frac{t}{a})$ where Ξ is a smooth function. This allows to conclude that for $\frac{t}{a} < c$ and a small enough, the Jacobian is nonzero. Hence t is not a conjugate time.

Moreover, we are going to prove that a geodesic with an initial condition $a > 0$ small enough intersects exactly one geodesic with an initial condition $\bar{a} < 0$ of the same length at $t \sim a$. Hence, the upper part of the cut locus generated by the geodesics corresponding to small a is exactly the set corresponding to the intersection of geodesics described below.

If two geodesics intersect at the same time, one with $a > 0$ and the other with $\bar{a} < 0$, then

$$a^2 + a^3 \psi'(0) + o(a^3) = \bar{a}^2 + \bar{a}^3 \psi'(0) + o(\bar{a}^3),$$

hence $\bar{a} = -a - a^2\psi'(0) + o(a^2)$, the cut time is $t_{cut} = a + \frac{1}{2}a^2\psi'(0) + o(a^2)$ and the cut point is

$$x_{cut} = -\frac{\psi'(0)}{2} a^2 + o(a^2), \quad y_{cut} = a^2 + o(a^2). \tag{5}$$

It is easy to see that when the two geodesics intersect, the corresponding fronts are transverse to each other whence the upper branch of the cut locus from $\mathcal{Z}$ is a smooth curve, in a sufficiently small neighborhood of $(0, 0)$. From (5) we deduce moreover that the tangent vector to $\mathbf{K}^+_{\mathcal{Z}}$ at $(0, 0)$ is $(-\psi'(0)/2, 1)$, which does not belong to the distribution at $(0, 0)$.

4.2 The lower part of the cut locus

Reasoning as in section 4.1, we consider the geodesic starting from a point of $\mathcal{Z}$ with initial condition

$$x(0) = a > 0,\ y(0) = a^2\psi(a),\ p_x(0) = 1,\ p_y(0) = -\frac{1}{2a\psi(a) + a^2\psi'(a)}, \tag{6}$$

i.e., the geodesic realizing locally the distance from $\mathcal{Z}$ and entering the lower domain. Taking $\eta = \sqrt{a}$, one can check that if x, y, p_x, p_y have orders in η higher or equal to $1, 3, 0, -2$, respectively, then the dynamics has the same or higher orders. As a consequence, since the initial condition respects these orders, one can compute jets with respect to η of the solution of system (2) under the form

$$\begin{array}{rclrcl} x(s) &=& \eta x_0(s) + \eta^2 x_1(s) + \eta^3 \bar{x}(\eta, s) & p_x(s) &=& p_{x0}(s) + \eta p_{x1}(s) + \eta^2 \bar{p}_x(\eta, s) \\ y(s) &=& \eta^3 y_0(s) + \eta^4 y_1(s) + \eta^5 \bar{y}(\eta, s) & p_y(s) &=& \eta^{-2} p_{y0}(s) + \eta^{-1} p_{y1}(s) + \bar{p}_y(\eta, s), \end{array}$$

where $\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y$ are smooth functions. From the initial condition (6), we deduce

$$\begin{array}{llll} x_0(0) = 0, & x_1(0) = 1, & p_{x0}(0) = 1, & p_{x1}(0) = 0, \\ y_0(0) = 0, & y_1(0) = 1, & p_{y0}(0) = -\frac{1}{2}, & p_{y1}(0) = 0, \end{array}$$

and using system (2), the functions $x_0, x_1, y_0, y_1, p_{x0}, p_{x1}, p_{y0}, p_{y1}$ satisfy

$$\left\{\begin{array}{ll} \dot{x}_0 &= p_{x0} \\ \dot{y}_0 &= \gamma^2 p_{y0} x_0^4 \\ \dot{p_{x0}} &= -2\gamma^2 p_{y0}^2 x_0^3 \\ \dot{p_{y0}} &= 0 \end{array}\right. \qquad \left\{\begin{array}{ll} \dot{x}_1 &= p_{x1} \\ \dot{y}_1 &= \gamma^2 (p_{y1} x_0^4 - 2p_{y0} x_0^2 (y_0 - 2x_0 x_1 - \alpha x_0^3)) \\ \dot{p_{x1}} &= \gamma^2 p_{y0} x_0 (-4p_{y1} x_0^2 + 2p_{y0} y_0 - 6p_{y0} x_0 x_1 - 5\alpha p_{y0} x_0^3) \\ \dot{p_{y1}} &= \gamma^2 p_{y0} x_0^2 \end{array}\right. \tag{7}$$

where $\gamma = \mathrm{e}^{\xi(0,0)}$ and $\alpha = \psi'(0) + \frac{\partial \xi}{\partial x}(0,0)$. Thus $p_{y0} \equiv -\frac{1}{2}$ and one can prove (see [2, 10]) that

$$\begin{array}{rcl} x_0(s) &=& -\dfrac{\sqrt{2}}{\sqrt{\gamma}} \mathrm{cn}\,(\mathcal{K} + \sqrt{\gamma} s), \\ y_0(s) &=& -\dfrac{2}{3\sqrt{\gamma}} (\sqrt{\gamma} s + 2\mathrm{sn}\,(\mathcal{K} + \sqrt{\gamma} s) \mathrm{cn}\,(\mathcal{K} + \sqrt{\gamma} s) \mathrm{dn}\,(\mathcal{K} + \sqrt{\gamma} s)), \end{array}$$

where $\mathcal{K}$ is the complete elliptic integral of the first kind of modulus $\frac{1}{\sqrt{2}}$, and cn, sn and dn denote the classical Jacobi functions of modulus $\frac{1}{\sqrt{2}}$. Recall that the Jacobi functions cn, sn are $4\mathcal{K}$-periodic, when dn is $2\mathcal{K}$-periodic.

Denote by $x_{10}, y_{10}, p_{x10}, p_{y10}$ the solution of the second system in (7) with $\alpha = 0$. Define g_1, g_2, g_3, g_4 by

$$\begin{array}{rclrcl} x_1 &=& x_{10} + \alpha g_1, & p_{x1} &=& p_{x10} + \alpha g_3, \\ y_1 &=& y_{10} + \alpha g_2, & p_{y1} &=& p_{y10} + \alpha g_4. \end{array}$$

It is easy to see that the g_i satisfy

$$\left\{\begin{array}{rcl} g_4 &\equiv& 0 \\ \dot{g}_1 &=& g_3 \\ \dot{g}_2 &=& -\gamma^2 x_0^3 (2g_1 + x_0^2) \\ \dot{g}_3 &=& -\frac{1}{4}\gamma^2 x_0^2 (6g_1 + 5x_0^2), \end{array}\right. \tag{8}$$

and the initial conditions are $g_1(0) = g_2(0) = g_3(0) = 0$. Notice moreover that, if

$$(x_0, y_0, p_{x0}, p_{y_0}, x_{10}, y_{10}, p_{x10}, p_{y_{10}}, g_1, g_2, g_3)$$

is the solution of (7), (8) with initial condition $(0, 0, 1, -1/2, 1, 1, 0, 0, 0, 0, 0)$ then the solution of (7), (8) with initial condition $(0, 0, -1, -1/2, -1, 1, 0, 0, 0, 0, 0)$ is

$$(-x_0, y_0, -p_{x0}, p_{y_0}, -x_{10}, y_{10}, -p_{x10}, p_{y_{10}}, g_1, -g_2, g_3),$$

which is a geodesic starting from a point of $\mathcal{Z}$ with $a < 0$.

Now, we are going to prove that it exists $\delta > 0$ such that if $\eta \neq 0$ is small enough and $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$ then, at t fixed, $\frac{\partial x(t)}{\partial a} > 0$. This implies in particular that two geodesics with initial conditions a and $\bar{a}$ of the same sign do not intersect at t if a and $\bar{a}$ are small enough and $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$.

In fact

$$\frac{\partial x(t)}{\partial \eta} = x_0(\frac{t}{\eta}) - \frac{t}{\eta}\dot{x}_0(\frac{t}{\eta}) + \eta(2x_1(\frac{t}{\eta}) - \frac{t}{\eta}\dot{x}_1(\frac{t}{\eta})) + \eta^2 x_r(\eta, \frac{t}{\eta})$$

where x_r is a smooth function. Now, the function $f : u \mapsto x_0(u) - u\dot{x}_0(u)$ is such that $f(0) = 0$ and

$$f'(u) = -u\ddot{x}_0(u) = \frac{1}{2}u\gamma^2 x_0^3(u) > 0 \text{ for } u \in]0, \frac{2\mathcal{K}}{\sqrt{\gamma}}[.$$

Hence, for ϵ small enough, it exists δ such that $f(u) > \epsilon$ for $u \in]\delta, \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta[$. Which implies that if η is small enough and $\delta < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$, then $\frac{\partial x(t)}{\partial \eta} > 0$. For $0 < \frac{t}{\eta} < \delta$ (possibly reducing δ and η), since $2x_1(\frac{t}{\eta}) - \frac{t}{\eta}\dot{x}_1(\frac{t}{\eta}) = 2$ for $t = 0$, we have that $\frac{\partial x(t)}{\partial \eta} > 0$.

This implies that two geodesics corresponding to initial conditions a and $\bar{a}$ of the same sign such that η is small enough cannot intersect at the time t satisfying $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$.

For what concerns the conjugate locus, one can compute that the Jacobian of $(\eta, s) \mapsto (x(s), y(s))$ is $\eta^3(J_0(s) + \eta J_1(s) + \eta^2 J_2(s, \eta))$, where $J_1(s) = x_0(s)\dot{y}_0(s) - 3y_0(s)\dot{x}_0(s)$, $J_1(0) = -4\text{sign}(\dot{x}_0(0))$, and J_2 is a smooth function. It was proven in [11] that J_0 is nonvanishing between 0 and $\bar{s}$ with $\bar{s}$ strictly bigger than $2\mathcal{K}/\sqrt{\gamma}$. Moreover $J_1(0)$ has the same sign as the function J_0 on the interval $]0, \bar{s}[$. This allows to conclude that exists $\delta > 0$ such that the Jacobian is nonvanishing on the interval $[0, \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta[$. This allows to conclude that if $\frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$ and a small enough, t is not a conjugate time.

Now, for an $a > 0$ small enough, we want to prove that there exists exactly one $\bar{a} < 0$ such that the geodesics starting with the initial conditions a and $\bar{a}$ are optimal until their intersection at a time t satisfying $0 < \frac{t}{\eta} < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta$. This will allow to conclude on the form of the local cut locus from $\mathcal{Z}$. In order to do that, we start by finding the good candidate.

For $a > 0$, the corresponding geodesic parametrized by s is

$$\begin{aligned} x_+(s) &= \eta x_0(s) + \eta^2(x_{10}(s) + \alpha g_1(s)) + o(\eta^2), \\ y_+(s) &= \eta^3 y_0(s) + \eta^4(y_{10}(s) + \alpha g_2(s)) + o(\eta^4), \end{aligned}$$

when for $\bar{a} < 0$ it is

$$\begin{aligned} x_-(s) &= -\eta x_0(s) + \eta^2(-x_{10}(s) + \alpha g_1(s)) + o(\eta^2), \\ y_-(s) &= \eta^3 y_0(s) + \eta^4(y_{10}(s) - \alpha g_2(s)) + o(\eta^4). \end{aligned}$$

Let us estimate these geodesics for $t_0 = \frac{2\mathcal{K}}{\sqrt{\gamma}}\eta_0$, $\eta_+ = \eta_0 + c_+\eta_0^2 + o(\eta_0^2)$ and $\eta_- = \eta_0 + c_-\eta_0^2 + o(\eta_0^2)$. One computes easily that

$$\begin{aligned} s_+ &= \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_+\eta_0 + o(\eta_0)), \\ s_- &= \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_-\eta_0 + o(\eta_0)), \end{aligned}$$

and

$$\begin{aligned} x_+(t_0) &= \eta_0^2(c_+\frac{2\mathcal{K}}{\sqrt{\gamma}} + x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + \alpha g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}})) + o(\eta^2), \\ y_+(t_0) &= -\eta_0^3\frac{4\mathcal{K}}{3\sqrt{\gamma}} + \eta_0^4(-\frac{4\mathcal{K}c_+}{\sqrt{\gamma}} + y_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + \alpha g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}})) + o(\eta^4), \\ x_-(t_0) &= \eta_0^2(-c_-\frac{2\mathcal{K}}{\sqrt{\gamma}} - x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + \alpha g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}})) + o(\eta^2), \\ y_-(t_0) &= -\eta_0^3\frac{4\mathcal{K}}{3\sqrt{\gamma}} + \eta_0^4(-\frac{4\mathcal{K}c_-}{\sqrt{\gamma}} + y_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}}) - \alpha g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}})) + o(\eta^4). \end{aligned}$$

Hence, these two geodesics intersect for

$$\begin{aligned} c_+ &= \sqrt{\gamma}\ \frac{\alpha g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) - 2x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{4\mathcal{K}}, \\ c_- &= -\sqrt{\gamma}\ \frac{\alpha g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + 2x_{10}(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{4\mathcal{K}}. \end{aligned}$$

The corresponding point is

$$\begin{aligned} x_{int}(t_0) &= \eta_0^2\alpha\ \frac{2g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}})}{2} + o(\eta^2), \\ y_{int}(t_0) &= -\eta_0^3\frac{4\mathcal{K}}{3\sqrt{\gamma}} + o(\eta^3), \end{aligned}$$

and the intersection time satisfies

$$\frac{t_0}{\eta_+} = \frac{2\mathcal{K}}{\sqrt{\gamma}}(1 - c_+\eta_+ o(\eta_+)) < \frac{2\mathcal{K}}{\sqrt{\gamma}} + \delta \tag{9}$$

for a small enough.

The inequality (9) proves that locally a geodesic cannot loose optimality by reaching the conjugate locus or by intersecting a geodesic with an initial condition a of the same sign.

We claim that the two geodesics are optimal until time t_0. The idea of the proof is that if one of the two geodesic looses optimality at $\bar{t} < t_0$, then there exists another geodesic optimal until $\bar{t}$ intersecting it at $\bar{t}$. But if this is the case, this new geodesic has lost optimality before by computation similar to the one giving rise to the inequality (9). As a consequence of these arguments, we can conclude that $(x_{int}(t_0), y_{int}(t_0))$ is a cut point.

One can check that $2g_1(\frac{2\mathcal{K}}{\sqrt{\gamma}}) + g_2(\frac{2\mathcal{K}}{\sqrt{\gamma}}) \neq 0$ which means that this lower part of the cut is half a cusp. Finally the cut is locally a curve since the fronts corresponding to $a > 0$ and $a < 0$ are transverse at the cut points.

5 Proof of Theorem 1 and of Proposition 6

In this section we prove the existence of a crest of the curvature passing through a tangency point q and satisfying the following conditions: **i)** it is smooth, **ii)** it has a tangent direction which is transverse to the distribution at q, **iii)** it admits a canonical parametrization.

Choose a coordinate system as in Section 4 for which we have a local representation satisfying $\psi(0) = 1$. By construction, K is well defined outside the singular set $\mathcal{Z}$.

The crests or valleys of K are implicitly defined by the equation

$$G(\nabla(||\nabla K||^2), (\nabla K)^\perp) = 0. \tag{10}$$

Computing the left hand side of equation (10), we find that

$$G(\nabla(||\nabla K||^2), (\nabla K)^\perp) = \frac{e^{2\xi(x,y)}h(x,y)}{(y - x^2\psi(x))^8},$$

where h is a smooth function. Hence, equation (10) is equivalent to $h(x, y) = 0$. The development of h at the point $(0, 0)$ is

$$h(x,y) = C\big(y^4(10x + y(3\psi'(0) - 2\xi_x(0,0))) + \sum_{i=0}^{6} a_i(x,y)x^i y^{6-i}\big),$$

where C is a nonzero constant and a_i are smooth functions. Let us show that there exists a smooth function $b : I \to \mathbf{R}$ defined on a neighborhood I of 0 such that after the coordinate change

$$X = 10x + y(3\psi'(0) - 2\xi_x(0,0)) - b(y)y^2, \quad Y = y,$$

we have $h(x(X,Y), y(X,Y)) = X\overline{h}(X,Y)$. In the new coordinate system, we have

$$h(x(X,Y), y(X,Y)) = C(Y^4X + F(X,Y)), \text{ where } F(X,Y) = \frac{Y^6}{10}b(Y) + \sum_{i=0}^{6} a_i(x(X,Y),Y)(x(X,Y))^i Y^{6-i}.$$

In order X to be factorizable in F, we require that $F(0,Y) \equiv 0$. Since $F(0,Y) = Y^6 R(b(Y), Y)$, where

$$R(b(Y),Y) = \frac{b(Y)}{10} + \sum_{i=0}^{6} \frac{a_i(x(0,Y),Y)}{10^i}(-3\psi'(0) + 2\xi_x(0,0) - b(Y)Y)^i,$$

it follows that $F(0,Y) \equiv 0$ if and only if there exists a smooth function b defined on a neighborhood of 0 such that $R(b(Y),Y) \equiv 0$. Let $\overline{b} = -10\sum_{i=0}^{6} \frac{a_i(0,0)}{10^i}(-3\psi'(0) + 2\xi_x(0,0))^i$. Then, since $R(b,Y)$ is smooth, $R(\overline{b}, 0) = 0$, $R_b(\overline{b}, 0) = 1/10$, we apply the Inverse Function Theorem to find a smooth function $b(Y)$ with the properties above. Therefore, coming back to the (x, y) coordinates we have shown that

$$h(x,y) = C(10x + y(3\psi'(0) - 2\xi_x(0,0)) + b(y)y^2)(y^4 + \tilde{F}(x,y)),$$

where $\tilde{F}$ is smooth and b is the function built above. The last equation implies that the curve $\{(x,y) \mid 10x + y(3\psi'(0) - 2\xi_x(0,0)) + b(y)y^2 = 0\}$ is a connected component of the set defined by equation (10), it is smooth, it passes through $(0, 0)$ and its tangent line at $(0, 0)$ is

$$x = \frac{1}{10}(2\xi_x(0,0) - 3\psi'(0))y,$$

that is transverse to the distribution at $(0,0)$.

We are left to find a canonical parametrization on the given curve. Notice that the limit of ∇K as (x,y) goes to $(0,0)$ does not exist, since the curvature does not converge at the tangency point. Nevertheless, it happens that if (x,y) tends to $(0,0)$ along a curve that approaches the origin with tangent direction $(2/10\xi_x(0,0)-3/10\psi'(0),1)$, then ∇K converges. Hence, we can choose the parametrization $s \mapsto \Gamma(s) = (x(s),y(s))$ such that $G(\nabla K, \dot\Gamma(s)) \equiv 1$, equivalently

$$-\partial_x K(x(s),y(s))(y(s)(3\psi'(0)-2\xi_x(0,0))+b(y(s))y(s)^2)+10\partial_y K(x(s),y(s))\dot y(s)=10.$$

Starting from the parametrized curve $s \mapsto \Gamma(s)$ and following the procedure in the proof of Lemma 1 in [3] (here we do not assume $\psi(0)=1$), we end up with a normal form (F3) where the functions ψ, ξ satisfy

$$2\xi_x(0,0)\psi(0)-3\psi'(0)=0.$$

■

6 Appendix

In this section we prove Proposition 3. Let $q \in M$ be such that $\nabla K(q) \neq 0$. In this case, the level set $\{p \in M \mid K(p) = K(q)\} \cap U$ is a smooth 1-dimensional submanifold of M. Using the local representation (F1), one gets

$$\nabla K(x,y) = \left(\partial_x^3\phi(x,y) - 2\partial_x\phi(x,y)\partial_x^2\phi(x,y), e^{2\phi(x,y)}\left(-2\partial_x\phi(x,y)\partial_x\partial_y\phi(x,y)+\partial_x^2\partial_y\phi(x,y)\right)\right).$$

Requiring that the level set of the curvature passing through q is the vertical axis, one gets that the second coordinate of $\nabla K(0,y)$ is zero. Hence we get

$$-2\partial_x\phi(0,y)\partial_x\partial_y\phi(0,y)+\partial_x^2\partial_y\phi(0,y) \equiv 0.$$

Requiring that the vertical axis is parameterized by arclength, one gets

$$\phi(0,y) \equiv 0.$$

Assume now that q is such that $\nabla K(q) = 0$. As in the previous section, we are looking for solutions to the equation

$$G(\nabla(||\nabla K||^2), (\nabla K)^\perp) = 0. \tag{11}$$

Consider an orthonormal frame of the type $(F1)$. The jet of the left hand side of equation (11) is

$$h(x,y) = h_0x^2 + h_1xy - h_0y^2 + \sum_{i=0}^{3} c_i(x,y)x^iy^{3-i},$$

where c_i are smooth functions and h_i are real numbers depending on the values of ϕ and its derivatives until order 4 at $(0,0)$. We study the generic case $(h_0,h_1) \neq (0,0)$. In this case h has a saddle in $(0,0)$ and the equation $h(x,y)=0$ defines locally two smooth curves which are respectively the crest and the valley of the curvature. Requiring that the vertical axis is a crest or a valley, we have $h_0 = 0$. Finally one can parameterize the crest by arclength. The explicit expression of h_0 is

$$
\begin{aligned}
h_0 &= 2e^{4\phi} \,(\, 4\partial_x^2\partial_y^1\phi^3\partial_x^3\phi - 4\partial_x^1\partial_y^2\phi\partial_x^2\partial_y\phi\partial_x^3\phi^2 + 8\partial_x\phi^3(\partial_x^2\partial_y\phi^3 - \partial_x\partial_y^2\phi\partial_x^2\partial_y\phi\partial_x^3\phi) - \\
&- 2\partial_x^2\phi^2\partial_x^2\partial_y^2\phi\partial_x^3\partial_y\phi + 2\partial_x\partial_y^2\phi\partial_x^2\phi\partial_x^3\phi\partial_x^3\partial_y\phi - \partial_x^3\partial_y\phi^3 + \partial_x^2\partial_y^2\phi\partial_x^3\partial_y\phi\partial_x^4\phi - \\
&- 2\partial_x\partial_y\phi^2\partial_x^3\partial_y\phi(4\partial_x^2\phi^2 + \partial_x^4\phi) + 4\partial_x\partial_y\phi^3(\partial_x^3\phi^2 + \partial_x^2\phi\partial_x^4\phi) + \\
&+ 2\partial_x\partial_y\phi(2\partial_x^2\phi^3\partial_x^2\partial_y^2\phi - 2\partial_x\partial_y^2\phi\partial_x^2\phi^2\partial_x^3\phi + 2\partial_x^2\partial_y^2\phi\partial_x^3\phi^2 - 5\partial_x^2\partial_y\phi\partial_x^3\phi\partial_x^3\partial_y\phi + \\
&+ 3\partial_x^2\phi\partial_x^3\partial_y\phi^2 + (\partial_x^2\partial_y\phi^2 - \partial_x^2\phi\partial_x^2\partial_y^2\phi + \partial_x\partial_y^2\phi\partial_x^3\phi)\partial_x^4\phi) + \\
&+ 4\partial_x\phi^2(-3\partial_x^2\partial_y\phi^2\partial_x^3\partial_y\phi + \partial_x\partial_y^2\phi\partial_x^3\phi\partial_x^3\partial_y\phi + \partial_x^2\partial_y\phi(\partial_x^2\partial_y^2\phi\partial_x^3\phi + \partial_x\partial_y^2\phi\partial_x^4\phi)) + \\
&+ \partial_x\phi(2\partial_x^3\partial_y\phi(3\partial_x^2\partial_y\phi\partial_x^3\partial_y\phi - \partial_x\partial_y^2\phi\partial_x^4\phi) - 2\partial_x^2\partial_y^2\phi(\partial_x^3\phi\partial_x^3\partial_y\phi + \partial_x^2\partial_y\phi\partial_x^4\phi)) \,) \,,
\end{aligned}
\tag{12}
$$

where all the derivatives of ϕ are computed at $(0, 0)$. ■

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