Two-Dimensional Almost-Riemannian Structures with Tangency Points

A.A. Agrachev, U. Boscain, G. Charlot, R. Ghezzi, M. Sigalotti

***Abstract*— Two dimensional almost-Riemannian geometries are metric structures on surfaces defined locally by a Lie bracket generating pair of vector fields. We study the relation between the topology of an almost-Riemannian structure on a compact oriented surface and the total curvature. In particular, we analyse the case in which there exist tangency points, i.e. points where two generators of the distribution and their Lie bracket are linearly dependent. The main result of the paper is a characterization of trivializable oriented almost-Riemannian structures on compact oriented surfaces in terms of the topological invariants of the structure. Moreover, we present a Gauss-Bonnet formula for almost-Riemannian structures with tangency points.**

I. Introduction

Let M be a two-dimensional smooth manifold. A Riemannian distance on M can be seen as the minimum-time function of an optimal control problem where admissible velocities are vectors of norm one. The control problem can be written locally as

$$\dot{q} = uX(q) + vY(q)\,, \quad u^2 + v^2 \leq 1\,, \tag{1}$$

by fixing an orthonormal frame (X, Y).

An almost-Riemannian structure generalizes a Riemannian one by allowing X and Y to be collinear at some points. If the pair (X, Y) is Lie bracket generating, i.e., if

$$\mathrm{span}\{X(q), Y(q), [X, Y](q), [X, [X, Y]](q), \ldots\} = T_q M$$

at every $q \in M$, then (1) is completely controllable and the minimum-time function still defines a continuous distance d on M. Notice that a Riemannian distance can be globally defined on M by a control problem as in (1) only if the Riemannian structure admits a global orthonormal frame X and Y, implying that M is parallelizable. More in general, X and Y are parallel on a set $\mathcal{Z} \subset M$ (called *singular locus*), which is generically a one-dimensional embedded submanifold of M (possibly disconnected). Metric structures that can be defined *locally* by a pair of vector fields (X, Y) through (1) are called *almost-Riemannian structures*. More precisely an almost-Riemannian structure can be seen as an atlas of local orthonormal frames

$$\mathcal{S} = \{(\Omega^\mu, X^\mu, Y^\mu)\}_{\mu \in I},$$

where $\{\Omega^\mu\}_{\mu\in I}$ is an open covering of M and, for every $\mu \in I$, (X^μ, Y^μ) is a pair of smooth vector fields defined on M, whose restriction to Ω^μ satisfies the Lie bracket generating condition. Moreover, for every $\mu, \nu \in I$ and for every $q \in \Omega^\mu \cap \Omega^\nu$, we assume that there exists an orthogonal matrix $R^{\mu,\nu}(q) = (R^{\mu,\nu}_{i,j}(q)) \in O(2)$ such that $X^\mu_i(q) = \sum_{j=1}^2 R^{\mu,\nu}_{i,j}(q) X^\nu_j(q)$. If each $R^{\mu,\nu}(q)$ belongs to $SO(2)$, we say that $\mathcal{S}$ is *orientable*. We say that an almost-Riemannian structure is *trivializable* if there exist X and Y such that $\mathcal{S} \cup \{(M, X, Y)\}$ is still an almost-Riemannian structure. The singular locus $\mathcal{Z}$ is the set of points of M at which the rank-varying distribution $q \mapsto \Delta(q) = \mathrm{span}\{X^\mu(q), Y^\mu(q) : \ q \in \Omega^\mu\}$ is one dimensional. An almost-Riemannian structure is Riemannian if and only if $\mathcal{Z} = \emptyset$.

The first example of genuinely almost-Riemannian structure is provided by the Grushin plane, which is the almost-Riemannian structure on $\mathbb{R}^2$ defined by the orthonormal frame $X(x, y) = (1, 0)$ and $Y(x, y) = (0, x)$. The model was originally introduced in the context of hypoelliptic operator theory [**?**], [**?**] (see also [**?**], [**?**]). Notice that the singular locus is indeed nonempty, being equal to the y-axis. Another example of (trivializable) almost-Riemannian structure has appeared in problems of control of quantum mechanical systems (see [**?**], [**?**]). In this case $M = S^2$ represents a suitable state space reduction of a three-level quantum system while the orthonormal generators X and Y are two infinitesimal rotations along two orthogonal axes, modeling the action on the system of two lasers.

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The notion of almost-Riemannian structure was introduced in [**?**]. In that paper, an almost-Riemannian structure is defined as a locally finitely generated Lie bracket generating $\mathcal{C}^\infty(M)$-submodule Δ of $\mathrm{Vec}(M)$, the space of smooth vector fields on M, endowed with a bilinear, symmetric map $G : \Delta \times \Delta \to \mathcal{C}^\infty(M)$ which is positive definite (in a suitable sense). A pair of vector fields (X, Y) in Δ is said to be orthonormal on some open set Ω if $G(X, Y)(q) = 0$ and $G(X, X)(q) = G(Y, Y)(q) = 1$ for every $q \in \Omega$. This definition is equivalent to the one given above in terms of an atlas of orthonormal frames.

This paper is a continuation of [**?**], where we provided a characterization of generic almost-Riemannian structures by means of local normal forms, and we proved a generalization of the Gauss-Bonnet formula. The main result consists of a characterization of trivializable generic oriented almost-Riemannian structures on compact oriented surfaces. Moreover, we generalize in a sense the Gauss-Bonnet formula of [**?**] to almost-Riemannian structures with tangency points.

The structure of the paper is the following. First of all, in Section II we recall some results contained in [**?**]. Then, in Section III the statement of the main result concerning the characterization of trivializable almost-Riemannian structures is given. Moreover, we outline the proof of the theorem by applying two lemmas. We then discuss the concept of integrability with respect to the Riemannian density induced by an oriented 2-ARS on the set of ordinary points. In particular, in Section IV-A we provide numerical simulations supporting the conjecture that, in presence of tangency points, the integral of the curvature defined in [**?**] as a limit of integrals does not converge. In Section IV-B we define the concept of *3-scale $\mathcal{S}$-integrability* which is useful to formulate the other result of the paper, which is a generalization of the Gauss-Bonnet formula given in [**?**] in presence of tangency points. As a direct consequence of the two results of the paper, in Section V we complete the analysis of the relation between the integral of the curvature and the topological invariants of an almost-Riemannian structure.

II. Preliminaries

The *flag* of a rank-varying distribution (M, Δ) is the sequence of submodules $\Delta = \Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_m \subset \cdots$ defined through the recursive formula

$$\Delta_{k+1} = \Delta_k + [\Delta, \Delta_k].$$

Denote by $\Delta_m(q)$ the set $\{V(q) \mid V \in \Delta_m\}$.

Under generic assumptions, the singular locus $\mathcal{Z}$ has the following properties: **(i)** $\mathcal{Z}$ is an embedded one-dimensional submanifold of M; **(ii)** the points $q \in M$ at which $\Delta_2(q)$ is one-dimensional are isolated; **(iii)** $\Delta_3(q) = T_qM$ for every $q \in M$.

We will say that $\mathcal{S}$ satisfies **(H0)** if properties **(i)**,**(ii)**,**(iii)** hold true. If this is the case, a point q of M is called *ordinary* if $\Delta(q) = T_qM$, *Grushin point* if $\Delta(q)$ is one-dimensional and $\Delta_2(q) = T_qM$, i.e. the distribution is transversal to $\mathcal{Z}$, and *tangency point* if $\Delta_2(q)$ is one-dimensional, i.e. the distribution is tangent to $\mathcal{Z}$. The set of ordinary points is $M \setminus \mathcal{Z}$, and if $\mathcal{S}$ satisfies **(H0)**, then the distribution is transversal to the singular locus at every point except for finitely many. Moreover, if (Ω, X, Y) is a local generator of Δ such that $\Omega \setminus \mathcal{Z}$ has exactly two components, then (X, Y) has different orientations on each of them.
The following proposition is a standard corollary of the transversality theorem. It formulates generic properties of a 2-ARS in terms of the flag of the distribution Δ.

*Proposition 1 ([**?**]):* Let M be a two-dimensional manifold. Then a 2-ARS on M generically satisfies condition **(H0)**.
ARSs satisfying hypothesis **(H0)** admit the following local normal forms.

*Theorem 1 ([**?**]):* Given a 2-ARS $\mathcal{S}$ satisfiyng **(H0)**, for every point $q \in M$ there exist a neighborhood U of q and an orthonormal frame (X, Y) on U such that up to a smooth change of coordinates defined on U, $q = (0, 0)$ and (X, Y) has one of the forms

(F1) $\quad X(x, y) = (1, 0), \quad Y(x, y) = (0, e^{\phi(x,y)}),$

(F2) $\quad X(x, y) = (1, 0), \quad Y(x, y) = (0, xe^{\phi(x,y)}),$

(F3) $\quad X(x, y) = (1, 0),$
$\quad Y(x, y) = (0, (y - x^2\psi(x))e^{\xi(x,y)}),$

where ϕ, ξ and ψ are smooth real-valued functions such that $\phi(0, y) = 0$ and $\psi(0) > 0$.
The main result of [**?**] is an extension of the Gauss-Bonnet theorem for orientable almost-Riemannian structures on orientable manifolds, under the hypothesis that there are not tangency points. More precisely, denote by $K : M \setminus \mathcal{Z} \to \mathbb{R}$ the Gaussian curvature. The first difficulty in order to extend the Gauss-Bonnet formula is to give a meaning to $\int_M K\, dA$, the integral of K on M with respect to the Riemannian density dA induced by the Riemannian metric on $M \setminus \mathcal{Z}$. The idea is to replace $K\, dA$ with a signed version of it. Fix an orientation on M and let M^+ (respectively M^-) be the subset of $M \setminus \mathcal{Z}$ were X^μ points on the right (resp. on the left) of Y^μ with respect to this orientation, and define $dA_s = dA$ on M^+ and $dA_s = -dA$ on M^-. The main goal of [**?**]

was to prove the existence and to assign a value to the limit

$$\lim_{\varepsilon\searrow 0}\int_{M_\varepsilon} K(q)dA_s, \tag{2}$$

where $M_\varepsilon = \{q \in M \mid d(q,\mathcal{Z}) > \varepsilon\}$ and $d(\cdot,\cdot)$ is the distance globally defined by the almost-Riemannian structure on M. We say that K is *$\mathcal{S}$-integrable* if the limit (2) exists and is finite. In this case we denote it by $\int_M KdA_s$. When $\mathcal{S}$ has no tangency points K happens to be $\mathcal{S}$-integrable and $\int_M KdA_s$ is determined by the topology of M^+ and M^-. This result, stated in Theorem 2 can be seen as a generalization of Gauss–Bonnet formula to ARSs.

*Theorem 2 ([**?**]):* Let M be a compact oriented two-dimensional manifold, endowed with an oriented 2-ARS $\mathcal{S}$ for which condition **(H0)** holds true. Assume that $\mathcal{S}$ has no tangency points. Then K is $\mathcal{S}$-integrable and $\int_M KdA_s = 2\pi(\chi(M^+) - \chi(M^-))$.

Once applied to the special subclass of Riemannian structures, such a result simply states that the integral of the curvature of a parallelizable compact oriented surface (i.e., the torus) is equal to zero. In a sense, in the standard Riemannian construction the topology of the surface gives a constraint on the total curvature through the Gauss-Bonnet formula, whereas for an almost-Riemannian structure induced by a single pair of vector fields the total curvature is equal to zero and the topology of the manifold constrains the metric to be singular on a suitable set.

III. Characterization of trivializable 2-ARS

Our objective is to complete the analysis in the more complicated case in which $\mathcal{S}$ has tangency points. The main result concerns the relation between the trivializability of an almost-Riemannian structure and the topological properties of the distribution and the singular locus associated to it. This theorem generalizes the condition $\chi(M^+) = \chi(M^-)$, which is necessary (see Lemma 5 of [**?**]) for an almost-Riemannian structure without tangency points to be trivializable, to the case in which tangency points exist.

Theorem 3: Let M be a compact oriented two-dimensional manifold endowed with an oriented almost-Riemannian structure $\mathcal{S}$ satisfying the generic hypothesis **(H0)**. Then $\mathcal{S}$ is trivializable if and only if $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$, where $\tau(\mathcal{S})$ is the number of rotations of Δ on $\mathcal{Z}$ computed with respect to the orientation induced by M^+ on $\mathcal{Z}$.

The integer $\tau(\mathcal{S})$ is defined as follows. Take a global notion of angle on TM, independent of $\mathcal{S}$. Such an angle can be induced by any fixed, globally defined, Riemannian metric g_0 on M. Consider an embedded closed surface $N \subset M$ with the induced orientation. Suppose ∂N is nonempty and Υ is a one-dimensional distribution on ∂N. We call *number of rotations* of Υ on a connected component W of ∂N the degree of any continuous (and therefore smooth) choice $\vartheta : W \to \mathbb{R}/\pi\mathbb{Z}$ of the g_0-angle between T_pW and $\Upsilon(p)$, computed with respect to the orientation of W induced by N. We denote it by $\tau_N(\Upsilon, W)$ and we define $\tau_N(\Upsilon, \partial N)$ as the sum of $\tau_N(\Upsilon, W)$ as W varies among the connected components of ∂N. Using this definition, we can associate to any oriented 2-ARS $\mathcal{S}$ and any connected component W of $\mathcal{Z}$ the integer $\tau_{M^+}(\Delta, W)$. We denote $\tau(\mathcal{S})$ the number $\tau_{M^+}(\Delta, \mathcal{Z})$.

As a direct consequence of Theorem 3 and of the simple remark that $\chi(M^+) + \chi(M^-) = \chi(M)$ is even we obtain that if $\mathcal{S}$ is trivializable then $\tau(\mathcal{S})$ is even.

The proof of the sufficiency of the condition $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$ is rather technical (for details see [**?**]). It consists of the construction of a globally defined vector field X such that $X \in \Delta$ and X has only non-degenerate zeros lying in $\mathcal{Z}$. First, we define X on a tubular neighborhood of $\mathcal{Z}$ in such a way that $X \in \Delta$ and has only a finite number of non-degenerate zeros lying in $\mathcal{Z}$, the algebraic number of zeros to be found later. After that, we extend X to the entire manifold. The hypothesis $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$ allows us to find an extension without introducing further zeros. This can be done by assigning to each connected component of $\mathcal{Z}$ a suitable integer representing the algebraic number of zeros of X on that component. Taking into account the topological contraints on the indeces of zeros of vector fields, to find these integers one has to solve a linear system where coefficients depend on the Euler characteristic of the connected components of $M \setminus \mathcal{Z}$ and on the number of rotations of Δ along each connected component of $\mathcal{Z}$.

The construction of a solution to this system is based on a correspondence between two-dimensional almost-Riemannian structures and bipartite graphs. Indeed, to an almost-Riemannian structure corresponds the bipartite graph whose vertices and edges are the connected components of $M \setminus \mathcal{Z}$ and $\mathcal{Z}$, respectively. Two vertices M_1 and M_2 are connected by an edge $\mathcal{Z}_{12}$ if and only if $\mathcal{Z}_{12} \subset \overline{M}_1 \cap \overline{M}_2$. The bipartite nature of the graph follows from the fact that a pair of vector fields generating Δ changes orientation while crossing $\mathcal{Z}$, so that $\mathcal{Z} = \overline{M^+} \cap \overline{M^-}$. There is a natural way of labeling vertices and edges of the graph associated with an almost-Riemannian structure. Indeed, we assign to each vertex M_l the number $\chi(M_l)$ (where χ denotes the Euler characteristic of manifolds) and to each edge $\mathcal{Z}_{ij}$ the integer $\tau_{M^+}(\Delta, \mathcal{Z}_{ij})$. The graph associated to $\mathcal{S}$ is invariant under the action of diffeomorphisms of M and this is why such graphs could be a useful instrument for a classification of almost-Riemanian structures.

The graph associated to the almost-Riemannian structure can be used to assign to each connected compontent of $\mathcal{Z}$ the suitable algebraic number of zeros of the vector field X. The existence of an extension of X with no further zeros is now a consequence of the choice made on the location of zeros and of the condition $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$.

Once a vector field $X \in \Delta$ with non-degenerate zeros lying in $\mathcal{Z}$ is constructed, we can assume it to have norm one by applying Lemma 1 below (for the proof see [**?**]).

Lemma 1: Let F be a vector field such that **i)** $F \in \Delta$, **ii)** its zeros are non-degenerate and lie in $\mathcal{Z}$. Then $|F| = \sqrt{G(F,F)}$ is a never vanishing smooth function. In particular, $\hat{F} = F/|F|$ is a well defined vector field of norm one.

Finally, to find a generator of Δ with norm one and orthogonal to X it is sufficient to rotate X in the sense of the following Lemma (see Lemma 4 in [**?**]).

Lemma 2: Let F_1 be a vector field such that **i)** $F_1 \in \Delta$, **ii)** $|F_1| = 1$. Let, moreover, $\theta : M \to \mathbb{R}/2\pi\mathbb{Z}$ be a smooth map. Then there exists a vector field F_2 such that $F_2 \in \Delta$, $|F_2| = 1$ and $G(F_1, F_2) = \cos\theta$ on M. In particular, when $\theta \equiv \pi/2$, (F_1, F_2) is an orthonormal frame of $\mathcal{S}$.

The proof of the necessity of the condition $\chi(M^+) - \chi(M^-) + \tau(\mathcal{S}) = 0$ is based on the existence of a global orthonormal frame whose elements have only non-degenerate zeros lying in $\mathcal{Z}$. Once a global orthonormal frame exists, one defines a suitable smooth angle $\varphi : M \to \mathbb{R}/2\pi\mathbb{Z}$ such that the rotation of the given frame in the sense of Lemma 2 by angle φ has the desired property. For more details see [**?**].

IV. $\mathcal{S}$-INTEGRABILITY IN PRESENCE OF TANGENCY POINTS

A. Numerical simulations

As concerns the notion of integrability of the curvature with respect to the Riemannian density on $M \setminus \mathcal{Z}$, it turns out that the hypothesis made in [**?**] about the absence of tangency points is not just technical. Indeed, in this section we provide some numerical simulations to support the conjecture that, in presence of tangency points,

$$\int_{M_\varepsilon} K dA_s$$

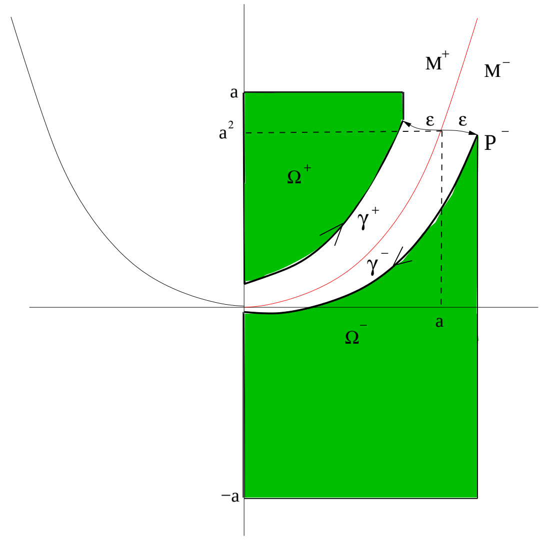

Fig. 1. Regions $\Omega^{\pm}$ where to apply Riemann Gauss-Bonnet formula

does not converge as ε tends to zero.

From the proof of Theorem 2 we know that far from tangency points the integral of geodesic curvature along ∂M_ε^+ and ∂M_ε^- offset each other for ε going to zero. Hence, to understand whether the presence of a tangency point may lead to non-$\mathcal{S}$-integrability of K it is sufficient to compute the geodesic curvature of ∂M_ε^+ and ∂M_ε^- in a neighborhood of such a point. More precisely consider the almost-Riemannian structure on the plane for which $(1, 0)$ and $(0, y - x^2)$ is an orthonormal frame. For this system one has $g = dx^2 + (y - x^2)^{-2} dy^2$, and

$$K = \frac{-2\left(3\,x^2 + y\right)}{\left(x^2 - y\right)^2}.$$

Notice that, in contrast with the behavior of the curvature in the Grushin plane (see [**?**]), in this case $\limsup_{q \to (0,0)} K(q) = +\infty$, while we still have $\liminf_{q \to (0,0)} K(q) = -\infty$. By symmetry reasons, for every $\varepsilon > 0$, the sets ∂M_ε^+ and ∂M_ε^- are smooth manifolds except at their intersections with the vertical axis $x = 0$, which is the cut locus for the problem of minimizing the distance from $\mathcal{Z} = \{(x, x^2) \mid x \in \mathbb{R}\}$. Fix $0 < a < 1$ and consider the two geodesics starting from the point (a, a^2) and minimizing (locally) the distance from $\mathcal{Z}$. Let P^+ and P^- be the two points along these geodesics at distance ε from $\mathcal{Z}$. Denote by γ^+ and γ^- the portions of ∂M_ε^+ and ∂M_ε^- connecting the vertical axis to the points P^+ and P^-, oriented as in Figure 1.

It is easy to approximate numerically γ^+ and γ^- by broken lines, but the evaluation of the integral

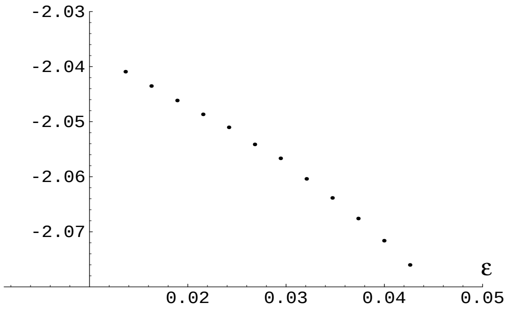

Fig. 2. Divergence of the $\mathcal{S}$-integral of K

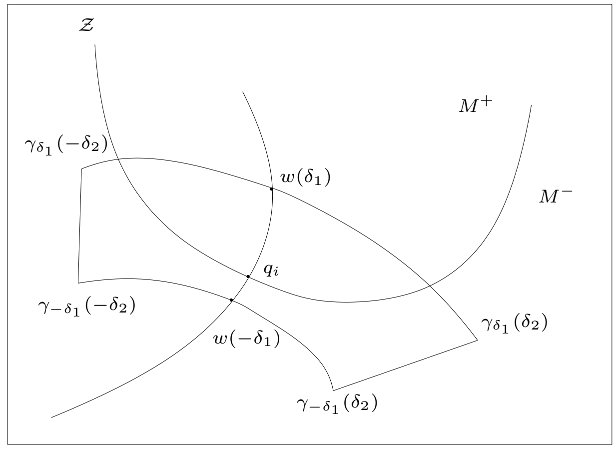

Fig. 3. The rectangular box $B^i_{\delta_1,\delta_2}$

of their geodesic curvatures is very unstable since its computation involves the second derivative of the curve parameterized by arclength. To avoid this problem, we rather apply the Riemannian Gauss-Bonnet formula on the regions Ω^+ and Ω^- introduced in Figure 1. This works better since the integral of the Gaussian curvature on Ω^+ and Ω^- is numerically stable, and the integral of the geodesic curvature on horizontal and vertical segments can be computed analytically (in particular it is always zero on horizontal segments).
Figure 2 shows the value of

$$\varepsilon\left(\int_{\gamma^+} K_g ds - \int_{\gamma^-} K_g ds\right)$$

for $a = 0.1$ and ε varying in the interval $[0.01, 0.04]$. The graph seems to converge as ε tends to zero to a nonzero constant, strongly hinting at the divergence of $\int_{M_\varepsilon} K dA_s$.

Similar pictures have been obtained for ARS on $\mathbb{R}^2$ generated by $(1,0)$, $(0, y-x^2-cx^3)$ with some constant c different from zero.

B. Gauss-Bonnet formula with tangency points

One possible explanation of the divergence of the limit (2) is the interaction between different orders in the asymptotic expansion of the almost-Riemannian distance. To avoid this interference, we define a 3-scale integral of the curvature. This can be done by introducing around each tangency point q_i, $i \in \{1,\dots,m_{\mathcal{S}}\}$, a two parameters "rectangle shaped" neighborhood $B^i_{\delta_1,\delta_2}$ (δ_1 and δ_2 playing the role of lengths of the sides of the rectangle) built as follows. We consider a parameterized curve $(-1,1) \ni s \mapsto w(s)$ passing through the tangency point $w(0) = q_i$ and transversal to the distribution. We then consider, for each $s \in (-1,1)$, the geodesic γ_s (parameterized by arclength) such that $\gamma_s(0) = w(s)$ and minimizing locally the distance from $\{w(s) : s \in (-1,1)\}$.

For δ_1, δ_2 sufficiently small, the rectangle $B^i_{\delta_1,\delta_2}$ is the subset of M containing the tangency point q_i and having as boundary (see Figure 3)

$$\gamma_{\delta_1}([-\delta_2,\delta_2]) \cup \gamma_{[-\delta_1,\delta_1]}(\delta_2) \cup \\ \cup \gamma_{-\delta_1}([-\delta_2,\delta_2]) \cup \gamma_{[-\delta_1,\delta_1]}(-\delta_2). \qquad (3)$$

Let $M_{\varepsilon,\delta_1,\delta_2} = M_\varepsilon \setminus \bigcup_{i=1}^{m_{\mathcal{S}}} B^i_{\delta_1,\delta_2}$. We say that K is *3-scale $\mathcal{S}$-integrable* (*3-$\mathcal{S}$-integrable* for short) if

$$\lim_{\delta_1\to 0}\lim_{\delta_2\to 0}\lim_{\varepsilon\to 0}\int_{M_{\varepsilon,\delta_1,\delta_2}} K dA_s \qquad (4)$$

exists and is finite. In this case we denote such a limit by $\oint_M K dA_s$. Notice that if $\mathcal{S}$ has not tangency points, then the concept of $\mathcal{S}$-integrability and 3-$\mathcal{S}$-integrability coincide. This more general notion of integrability is used in next theorem which generalizes Theorem 2.

Theorem 4: Let M be a compact oriented two-dimensional manifold. For an oriented almost-Riemanian structure on M satisfying the generic hypothesis **(H0)**, K is 3-scale $\mathcal{S}$-integrable and

$$\oint_M K dA_s = 2\pi(\chi(M^+) - \chi(M^-) + \tau(\mathcal{S})).$$

The order in which the limits are taken is important. Indeed, if the order is permuted, then the result given in Theorem 4 does not hold anymore. Recall that the normal form (F3) is not totally intrinsic, since the functions ψ and ϕ depend on the choice of a parametrized smooth curve passing through the tangency point q_0 and transversal to $\Delta(q_0)$. However, the result given in Theorem 4 does not depend on this choice. An interesting question is whether a canonical way of choosing these manifolds and their parameterizations exists. This is related to the problem of finding intrinsic normal forms for Grushin and tangency points.

The proof of Theorem 4 (for details see [**?**]) is based on a result given in [**?**] which is a Gauss-Bonnet formula for two-dimensional almost-Riemannian structures on admissible domains with boundary.

V. Conclusions

Theorems 3 and 4 allow us to complete the analysis between the integral of the curvature and the topology of the manifold for two-dimensional almost-Riemannian structures (see Corollary 1 in [**?**]). Indeed, as a consequence of Theorems 3 and 4 we get the following corollary.

Corollary 1: Let M be a compact oriented two-dimensional manifold. For an oriented almost-Riemannian structure $\mathcal{S}$ on M satisfying the generic hypothesis **(H0)** we have $⨍_M K dA_s = 0$ if and only if $\mathcal{S}$ is trivializable.

In particular, if $\mathcal{S}$ has not tangency points then $\int_M K dA_s = 0$ if and only if $\mathcal{S}$ is trivializable.